Module - 5

5.1. Rotation motion

In rotation motion all the particles of a rigid body move in concentric circular path about a fixed point or axis. The motion of pulley, shaft, flywheel etc are some of the examples of rotation motion.

5.2. Kinematics of rotation

When a rigid body rotates about a fixed axis, its position at any instant can be defined by angular displacement of a certain plane of the body passing through the axis of rotation. This angle of rotation is generally denoted by θ and is expressed in radians. When the body turns such that in equal intervals of time it describes equal angle of rotation, the motion is said to be uniform rotation.

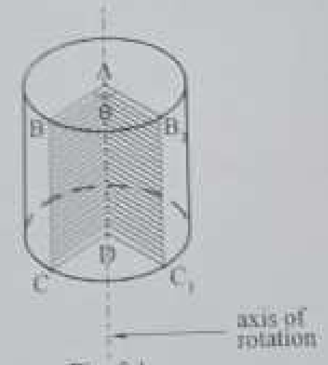

Fig. 5.1

The displacement of the particle in circular motion is measured in terms of angular displacement, θ, where θ is in radian. When a particle moves from position A to B along an arc of radius r, the angular displacement is θ as shown in Fig.5.2. The rate of change of angular displacement with time is called angular velocity and is denoted by ω.

$$\omega = \frac{d\theta}{dt} \text{ rad/s}$$

The rate of change of angular velocity with time is called angular acceleration and is denoted by α.

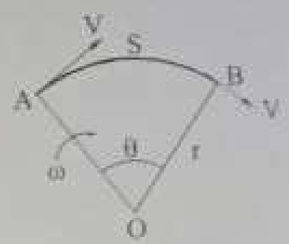

Fig. 5.2

$$\alpha = \frac{d\omega}{dt}$$

$$= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2} \text{ rad/s}^2$$

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \frac{d\omega}{d\theta}\,\omega$$

$$\alpha = \omega \frac{d\omega}{d\theta}$$

Sometimes angular velocity is expressed in terms of revolutions per minute (rpm). Since there are 2π radians in one revolution and 60 seconds in one minute, the angular velocity ω is given by

$$\omega = \frac{2\pi N}{60} \text{ rad/s}$$, where N is the angular velocity in rpm.

If the angular velocity is uniform the angular displacement in 't' seconds,

$$\theta = \omega t \text{ radians.}$$

When the angular accelaration is uniform,

$$\frac{d\omega}{dt} = \alpha$$

$$d\omega = \alpha\, dt$$

$$\omega = \int \alpha\, dt = \alpha t + C_1$$

where C_1, is the constant of integration.

If at $t = 0$, $\omega = \omega_0$, then

$$\omega_0 = \alpha \times 0 + C_1$$

$$\therefore C_1 = \omega_0$$

$$\omega = \alpha t + \omega_0$$

$$\omega = \omega_0 + \alpha t$$

$$\frac{d\theta}{dt} = \omega$$

$$d\theta = \omega\, dt$$

$$= (\omega_0 + \alpha t)\, dt$$

$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 + C_2$, where C_2 is the constant of integration

at $t = 0, \theta = 0$

$$\therefore 0 = 0 + 0 + C_2$$

$$C_2 = 0$$

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$$

$$\alpha = \frac{d\omega}{dt}$$

$$= \frac{d\omega}{d\theta} \frac{d\theta}{dt}$$

$$= \omega \frac{d\omega}{d\theta}$$

$$\alpha\, d\theta = \omega\, d\omega$$

Integrating $\alpha\theta = \frac{\omega^2}{2} + C_3$ is the constant of integration.

at $\theta = 0, \omega = \omega_0$

$$\alpha \times 0 = \frac{\omega_0^2}{2} + C_3$$

$$C_3 = -\frac{\omega_0^2}{2}$$

$$\alpha\theta = \frac{\omega^2}{2} - \frac{\omega_0^2}{2}$$

$$\omega^2 - \omega_0^2 = 2\alpha\theta$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

Thus for uniformly accelerated angular motion,

$$\omega = \omega_0 + \alpha t$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

For uniform angular retardation, use the following equations,

$$\omega = \omega_0 - \alpha t$$

$$\omega^2 = \omega_0^2 - 2\alpha\theta$$

$$\theta = \omega_0 t - \frac{1}{2}\alpha t^2$$

Analogy between angular and rectilinear motion.

Following are the analogy between angular and rectilinear motions.

Particulars	Angular motion	Rectilinear motion.
Initial velocity	ω_0	u
Final velocity	ω	V
Uniform Acceleration	α	a
Formula for velocity	$\omega = \frac{d\theta}{dt}$	$V = \frac{dS}{dt}$
Formula for acceleration	$\alpha = \frac{d\omega}{dt}$	$a = \frac{dV}{dt}$
Displacement	θ	S
Equations of motion	$\omega = \omega_0 + \alpha t$	$V = u + at$
	$\omega^2 = \omega_0^2 + 2\alpha\theta$	$V^2 = u^2 + 2aS$
	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$S = ut + \frac{1}{2}at^2$

Relation between linear velocity and angular velocity

Consider the motion of a particle along the circular path of radius r. The linear displacement of particle in 't' seconds is AB and angular displacement is θ as shown in Fig.5.3.

Linear displacement, AB = S = rθ

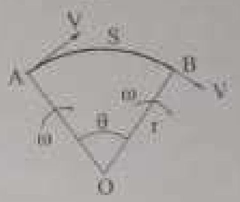

Fig. 5.3

$$V = \frac{dS}{dt} = r\frac{d\theta}{dt} = r\omega$$

$$V = r\omega.$$

Relation between linear acceleration and angular acceleration

Rate of change of velocity,

$$a = \frac{dV}{dt} = \frac{d}{dt}(r\omega)$$

$$= r\frac{d\omega}{dt}$$

$a = r\alpha$, a is the linear acceleration and α is the angular acceleration of the particle.

Example 5.1

The armature of an electric motor, has angular speed of 1800 rpm at the instant when the power is cut off. If it comes to rest in 6 seconds, calculate the angular deceleration assuming it is constant. How many revolutions does the armature make during this period.

Solution:

$$N_1 = 1800 \text{ rpm}$$

$$\omega_1 = \frac{2\pi N_1}{60} = 2\pi\times 30 \text{ rad/s}$$

$$\omega_2 = \omega_1 + \alpha t$$

$$0 = 2\pi\times 30 - \alpha\times 6$$

$$\alpha = \frac{2\pi\times 30}{6} = 31.4 \text{ rad/s}^2$$

Angle turned, $\theta = \omega_1 t - \frac{1}{2}\alpha t^2$

$$= 2\pi \times 30 \times 6 - \frac{1}{2} \times 31.4 \times 6^2$$

$$= 565.77 \text{ rad}$$

$$= 90 \text{ revolutions}$$

Example 5.2

A flywheel rotates with a constant retardation due to brakeing. In the first 10 seconds, it made 300 revolutions. At t = 7.5 s, its angular veocity was 40π rad/s. Determine,

(i) the value of constant retardation

(ii) the total time taken to come to rest and

(iii) the total revolutions made till it comes to rest.

Solution:

Angular displacement $\theta = \omega_1 t + \frac{1}{2}\alpha t^2$

At t = 10, $\theta = 300 \times 2\pi$ rad

$$300 \times 2\pi = \omega_1 \times 10 + \frac{1}{2} \times 10^2 \times \alpha$$

$$60\pi = \omega_1 + 5\alpha \text{ ....(i)}$$

$$\omega_2 = \omega_1 + \alpha t$$

at t = 7.5s, $\omega = 40\pi$ rad/s

$$40\pi = \omega_1 + 7.5\alpha \text{ ....(ii)}$$

From equations (i) and (ii)

$$\alpha = -8\pi \text{ rad/s}^2$$

From equation (i),

$$60\pi = \omega_1 + 5 \times (-8\pi)$$

$$\omega_1 = 100\pi \text{ rad/s}$$

When the flywheel comes to rest, $\omega = 0$

$\omega_1 = 100\pi$, $\omega_2 = 0$, $\alpha = -8\pi$ rad/s^2

$$\omega_2 = \omega_1 + \alpha t$$

$$\theta = 100\pi + (-8\pi) \times t$$

Total time taken to come to rest $t = \frac{100\pi}{8\pi}$

$$= 12.5\ s$$

Total revolutions made by the flywheel before it comes to rest,

$$\omega_2^2 = \omega_1^2 + 2\alpha\theta$$

$$0 = (100\pi)^2 + 2 \times (-8\pi)\theta$$

$$\theta = 19625\ rad$$

$$= 312.5\ revolutions$$

Example 5.3

A rotor of an electric motor is uniformly accelerated to a speed of 1800 rpm from rest in 5 seconds and then immediately power is switched off and the rotor decelerates uniformly to rest. If the total time elapsed is 12.5 s, determine the number of revolutions made in (i) acceleration and (ii) deceleration.

Solution:

The rotor is accelerated from rest

$$\therefore \omega_1 = 0$$

$$\omega_2 = \frac{2\pi N_2}{60} = \frac{2\pi \times 1800}{60}$$

$$= 60\ \pi\ rad/s$$

$$\omega_2 = \omega_1 + \alpha_{1-2} \times t_{1-2}$$

$$60\ \pi = 0 + 5\alpha_{1-2}$$

$$\alpha_{1-2} = 12\ \pi\ rad/s^2$$

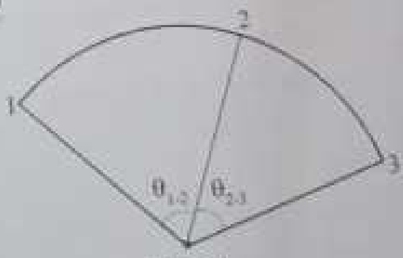

Fig. 5.4

$$\theta_{1-2} = \omega_1 t_{1-2} + \frac{1}{2}\alpha_{1-2} \times t^2_{1-2}$$

$$= 0 + \frac{1}{2} \times 12\pi \times 5^2$$

$$= 150\ \pi\ rad$$

$= 75$ revolutions

$$\omega_3 = 0$$

$$\omega_3 = \omega_2 + \alpha_{2-3} \times t_{2-3}$$

Total time of rotation is 12.5s

$$\therefore t_{2-3} = 12.5 - 5 = 7.5\text{ s}$$

$$0 = 60\,\pi + \alpha_{2-3} \times 7.5$$

$$\alpha_{2-3} = -8\,\pi\text{ rad/s}^2$$

$$\theta_{2-3} = \omega_2 \times t_{2-3} + \frac{1}{2}\alpha_{2-3} \times t_{2-3}^2$$

$$= 60\,\pi \times 7.5 + \frac{1}{2}(-8\,\pi) \times 7.5^2$$

$$= 450\,\pi - 225\,\pi$$

$$= 225\,\pi$$

$= 112.5$ revolutions

Example 5.4

A grinding wheel is attached to the shaft of an electric motor of rated speed of 1800 rpm. When the power is switched on the unit attains the rated speed in 5 second, and when the power is switched off the unit comes to rest in 90 seconds. Assuming uniformly accelerated motion, determine the number of revolutions the unit turns,

(i) to attain the rated speed and

(ii) to come to rest

Solution:

Initial angular velocity $\omega_1 = 0$

After 5 seconds the angular velocity,

$$\omega_2 = \frac{2\pi \times 1800}{60}$$

$$= 60\,\pi\text{ rad/s}$$

$$\omega_2 = \omega_1 + \alpha\,t$$

$$60\pi = 0 + \alpha \times 5$$

$$\alpha = 12\pi$$

Angular displacement, $\theta = \omega_1 t + \frac{1}{2} \times \alpha \times t^2$

$$= 0 + \frac{1}{2} \times 12\pi \times 5^2$$

$$= 150\pi \text{ rad}$$

$\therefore$ No. of revolutions to attain the rated speed $= \frac{150\pi}{2\pi} = 75$

When the power is switched off the unit comes to rest in 90 seconds.

$$\omega_2 = 60\pi \text{ rad/s}$$

$$\omega_3 = 0$$

$$\text{time } t = 90 \text{ s}$$

$$\omega_3 = \omega_2 + \alpha t$$

$$0 = 60\pi + \alpha \times 90$$

$$\alpha = -\frac{2}{3}\pi \text{ rad/s}$$

Angular rotation during this 90 seconds,

$$\theta = \omega_2 t + \frac{1}{2}\alpha \times t^2$$

$$= 60\pi \times 90 + \frac{1}{2}\left(-\frac{2}{3}\pi\right) \times 90^2$$

$$= 2700\pi \text{ rad}$$

$$= 1350 \text{ revolutions}$$

Example 5.5

A wheel accelerates from rest to a speed of 180 rpm uniformly in 0.4 seconds. It then rotates at that speed for 2 seconds before decelerating to rest in 0.3 seconds. Determine the total revolutions made by the wheel.

Solution:

Since the wheel accelerates from rest, $\omega_1 = 0$

$N_2 = 180$ rpm

$$\omega_2 = \frac{2\pi N_2}{60} = \frac{2\pi \times 180}{60} = 6\pi \text{ rad/s}$$

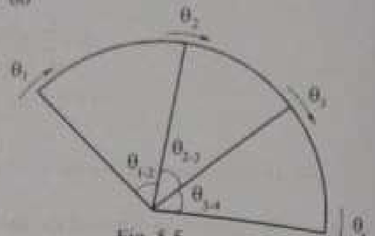

Fig. 5.5

$t_{1-2} = 0.4s$

$t_{2-3} = 2s$

$t_{3-4} = 0.3s$

$\omega_2 = \omega_1 + \alpha t_{1-2}$

$6\pi = 0 + \alpha \times 0.4$

$\alpha = 47.1 \text{ rad/s}^2$

$$\theta_{1-2} = \omega_1 t_{1-2} + \frac{1}{2}\alpha t_{1-2}^2$$

$$= 0 + \frac{1}{2} \times 47.1 \times 0.4^2$$

$= 3.77$ rad

During θ_{2-3} rotation of the wheel, the angular acceleration is zero.

$\therefore \omega_3 = \omega_2$

$\theta_{2-3} = \omega_2 \times t_{2-3}$

$= 6\pi \times 2$

$= 12\pi$ rad

$\omega_4 = \omega_3 + \alpha \times t_{3-4}$

$0 = 6\pi + \alpha \times 0.3$

$\alpha = -62.8 \text{ rad/s}^2$

$$\theta_{3-4} = \omega_3 t_{3-4} + \frac{1}{2}\alpha t_{3-4}^2$$

$$= 6\pi \times 0.3 - \frac{1}{2} \times 62.8 \times 0.3^2$$

$= 2.83$ rad

Total angular displacement

$\theta = \theta_{1-2} + \theta_{2-3} + \theta_{3-4}$

$= 3.77 + 12\pi + 2.83$

$= 44.28$ rad

$$= \frac{44.28}{2\pi}$$

$= 7.05$ revolutions

Example 5.6

A wheel rotates for 5 seconds with a constant angular acceleration and describes during that time 100 radian. It then rotates with constant angular velocity and during the next 5 second describes 80 radian. Find the initial angular velocity and the angular acceleration.

Solution:

Given: $\theta_{1-2} = 100$ rad $\quad \theta_{2-3} = 80$ rad

$t_{1-2} = 5$ s $\quad t_{2-3} = 5$ s $\quad \alpha_{2-3} = 0$

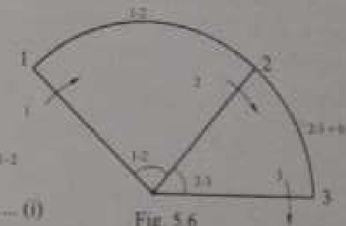

Fig. 5.6

$\omega_2 = \omega_1 + \alpha_{1-2} \times t_{1-2}$

$\omega_2 = \omega_1 + 5\alpha_{1-2}$ ..... (i)

$$\theta_{1-2} = \omega_1 t_{1-2} + \frac{1}{2} \times \alpha_{1-2} \times t_{1-2}^2$$

$$100 = 5\omega_1 + \frac{1}{2} \times \alpha_{1-2} \times 5^2$$

$20 = \omega_1 + 2.5\alpha_{1-2}$ ..... (ii)

$$\theta_{2-3} = \omega_2 \times t_{2-3} + \frac{1}{2} \times \alpha_{2-3} \times t_{2-3}^2$$

$80 = 5\omega_2 + 0$

$$\omega_2 = 16 \text{ rad/s}$$

From equation (i)

$$16 = \omega_1 + 5\alpha_{1-2}$$

From equation (ii)

$$20 = \omega_1 + 2.5\alpha_{1-2}$$

$$-4 = 2.5\alpha_{1-2}$$

$$\alpha_{1-2} = -1.6 \text{ rad/s}^2$$

$$16 = \omega_1 + 5\alpha_{1-2}$$

$$16 = \omega_1 - 5 \times 1.6$$

$$\omega_1 = 24 \text{ rad/s}$$

Example 5.7

A shaft is uniformly accelerated from 10 rev/s to 18 rev/s in 4 seconds. The shaft continues to accelerate at this rate for the next 5 seconds. Thereafter the shaft rotates with a uniform angular speed. Find the total time to complete 400 revolutions.

Solution:

$$\omega_1 = 10 \text{ rev/s}$$

$$= 2\pi \times 10 = 20\pi \text{ rad/s}$$

$$\omega_2 = 18 \text{ rev/s} = 36\pi \text{ rad/s}$$

$t_{1-2} = 4\,s$, $t_{2-3} = 5s$

Total angular rotation is $= 400 \times 2\pi$

$$= 800\pi \text{ rad}$$

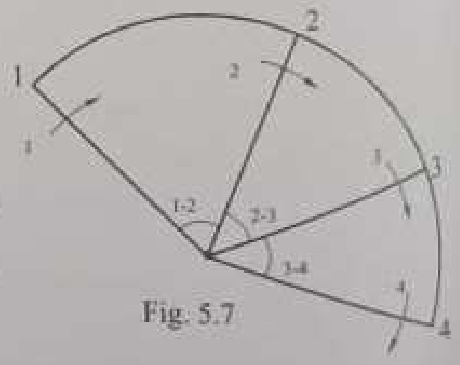

Fig. 5.7

$$\omega_2 = \omega_1 + \alpha_{1-2} \times t_{1-2}$$

$$36\pi = 20\pi + 4 \times \alpha_{1-2}$$

$$\alpha_{1-2} = 4\pi \text{ rad/s}^2$$

$$\theta_{1-2} = \omega_1 \times t_{1-2} + \frac{1}{2}\alpha_{1-2} \times t_{1-2}^2$$

$$= 20\,\pi \times 4 + \frac{1}{2} \times 4\,\pi \times 4^2$$

$$= 80\,\pi + 32\,\pi = 112\,\pi \text{ rad}$$

$$\theta_{2-3} = \omega_2\, t_{2-3} + \frac{1}{2} \times \alpha_{2-3} \times t_{2-3}^2$$

$$= 36\,\pi \times 5 + \frac{1}{2} \times 4\,\pi \times 5^2$$

$$= 180\,\pi + 50\,\pi$$

$$= 230\,\pi \text{ rad}$$

$$\omega_3 = \omega_2 + \alpha_{2-3} \times t_{2-3}$$

$$= 36\,\pi + 4\pi \times 5$$

$$= 56\,\pi \text{ rad/s}$$

$$\theta_{3-4} = \omega_3 \times t_{3-4} + \frac{1}{2} \times \alpha_{3-4} \times t_{3-4}^2, \quad \alpha_{3-4} = 0$$

$$= \omega_3 \times t_{3-4}$$

$$= 56\pi \times t_{3-4}$$

$$\theta_{3-4} = \text{Total angular displacement} - [\theta_{1-2} + \theta_{2-3}]$$

$$= 800\,\pi - [112\,\pi + 230\,\pi]$$

$$= 458\,\pi \text{ rad}$$

$$458\,\pi = 56\,\pi \times t_{3-4}$$

$$t_{3-4} = 8.18 \text{ s}$$

Total time to complete 400 revolutions is $4 + 5 + 8.18$

$$= 17.18 \text{ s}$$

Example 5.8

A slender semi circular wire of radius r is supported in its own vertical plane by a hinge at O and a smooth peg at A as shown in Fig 5.8. If the peg starts from O and moves with constant speed V_0 along the horizontal x axis, find the angular velocity of wire at the instant when $\theta = 60°$.

Solution:

As the peg A moves forward, the wire rotates about O in counter clockwise direction.

The angle OAB is always 90°.

$$OB = 2r$$

$$OA = 2r \sin\theta$$

$$V_0 t = 2r \sin\theta$$

$$\sin\theta = \frac{V_0 t}{2r}$$

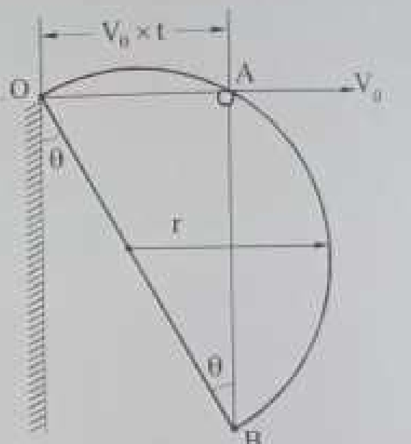

Fig. 5.8

Differentiating with respect to time,

$$\cos\theta \times \frac{d\theta}{dt} = \frac{V_0}{2r}$$

$$\cos\theta \times \omega = \frac{V_0}{2r}$$

$$\omega = \frac{V_0}{2r\cos\theta}$$

When $\theta = 60°$

$$\omega = \frac{V_0}{2r \times \cos 60}$$

$$= \frac{V_0}{r}$$

Example 5.9

The angle of rotation of a body is given as a function of time by the equation $\theta = \theta_0 + bt + ct^2$. Find general expressions for the angular veocity and angular acceleration of the body. Determine also the values of the constant b and c if the initial angular velocity is 2π rad/s and 1 second later it is 4π rad/s.

Solution:

Angular displacement, $\theta = \theta_0 + bt + ct^2$

Angular velocity, $\omega = \frac{d\theta}{dt} = 0 + b + 2ct$

$= b + 2ct$

at $t = 0$, $\omega = 2\pi$ rad/s

$2\pi = b + 0$

$b = 2\pi$ rad/s

at $t = 1$, $\omega = 4\pi$ rad/s

$4\pi = b + 2c \times 1$

$= 2\pi + 2c$

$c = \pi$ rad/s^2

$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}[b + 2ct]$

$= 2c$

$= 2\pi$ rad/s^2

Example 5.10

The rotation of a flywheel is defined by the equation, $\omega = 3t^2 - 2t + 2$, where ω is in rad/s and t is in seconds. After one second from the start, the angular displacement was 4 radians. Determine the angular displacement, angular velocity and angular acceleration of the flywheel when t = 3s.

Solution:

$\omega = 3t^2 - 2t + 2$

$\frac{d\theta}{dt} = 3t^2 - 2t + 2$

$d\theta = (3t^2 - 2t + 2)dt$

Integrating, $\int d\theta = \int (3t^2 - 2t + 2)dt$

$$\theta = \frac{3t^3}{3} - \frac{2t^2}{2} + 2t + c$$

$= t^3 - t^2 + 2t + c$ where c is the constant of integration.

at t = 1 s, θ = 4 rad

$$4 = 1^3 - 1^2 + 2\times 1 + c$$

$$c = 2$$

Angular displacement after 3 seconds

$$\theta = 3^3 - 3^2 + 2\times 3 + 2$$

$$= 26 \text{ rad}$$

Angular velocity after 3 seconds,

$$\omega = 3\times 3^2 - 2\times 3 + 2$$

$$= 23 \text{ rad/s}$$

Angular acceleration, $\alpha = \frac{d\omega}{dt}$

$$= \frac{d}{dt}\left[3t^2 - 2t + 2\right]$$

$$= 6t - 2$$

Angular acceleration after 3 seconds,

$$\alpha = 6\times 3 - 2$$

$$= 16 \text{ rad/s}^2$$

Example 5.11

The angular acceleration of a wheel is given by $\alpha = 12 - t$ where α is in rad/s² and t is in seconds. If the angular velocity of the wheel is 60 rad/s at the end of 4 seconds, determine the angular velocity at the end of 6 seconds. Calculate the number of revolutions in these 6 seconds.

Solution:

$$\alpha = 12 - t$$

$$\frac{d\omega}{dt} = 12 - t$$

$$d\omega = (12 - t)\,dt$$

Integrating, $\omega = 12t - \frac{t^2}{2} + c_1$, c_1 is the constant of integration.

at $t = 4s$, $\omega = 60$ rad/s

$$60 = 12 \times 4 - \frac{4^2}{2} + c_1$$

$$c_1 = 20$$

Angular velocity, $\omega = 12t - \frac{t^2}{2} + 20$

Angular velocity at the end of 6 seconds,

$$\omega = 12 \times 6 - \frac{6^2}{2} + 20$$

$$= 74 \text{ rad/s}$$

Angular velocity, $\omega = 12t - \frac{t^2}{2} + 20$

$$\frac{d\theta}{dt} = 12t - \frac{t^2}{2} + 20$$

$$d\theta = \left(12t - \frac{t^2}{2} + 20\right) dt$$

Integrating, $\theta = \frac{12t^2}{2} - \frac{t^3}{6} + 20t + c_2$

$$= 6t^2 - \frac{t^3}{6} + 20t + c_2$$

Let at $t = 0$, θ be θ_0

$$\theta_0 = 0 - 0 + 0 + c_2$$

$$c_2 = \theta_0$$

Angular displacement in 6 seconds,

$$\theta = 6t^2 - \frac{t^3}{6} + 20t + \theta_0$$

$$\theta - \theta_0 = 6t^2 - \frac{t^3}{6} + 20t$$

$$= 6\times 6^2 - \frac{6^3}{6} + 20\times 6$$

$$= 300 \text{ rad}$$

$$= \frac{300}{2\pi} \text{ revolutions}$$

$$= 47.77 \text{ revolutions}$$

Example 5.12

A wheel rotating about a fixed axis at 24 rpm is uniformly accelerated for 70 seconds, during which it makes 50 revolutions. Find

(i) angular velocity at the end of this interval and

(ii) the time required for the speed to reach 150 rpm.

Solution.

$$\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi \times 24}{60} = 2.51 \text{ rad/s}$$

$$t = 70 \text{ s}$$

$$\theta = 50 \text{ revolution} = 50 \times 2\pi \text{ rad} = 314 \text{ rad}$$

(i) Angular velocity after 70 s

$$\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

$$314 = 2.51 \times 70 + \frac{1}{2} \times \alpha\, 70^2$$

$$\alpha = 0.056 \text{ rad/s}^2$$

$$\omega_2 = \omega_1 + a\,t$$

$$= 2.51 + 0.056 \times 70$$

$$= 6.43 \text{ rad/s}$$

(ii) Time required to reach 150 rpm.

$$\omega_2 = \frac{2\pi N}{60} = \frac{2\pi \times 150}{60} = 15.71 \text{ rad/s}$$

$$\omega_2 = \omega_1 + \alpha t$$

$$15.71 = 2.51 + 0.056 \times t$$

$$t = 235.71 \text{ s.}$$

Example 5.13.

A body accelerates uniformly at 5 rad/s^2 is found to be rotating at 90 rad/s at the end of 12 seconds. Determine the initial velocity and the angle turned during this interval.

Solution.

$$\alpha = 5 \text{ rad/s}^2, \ \omega_2 = 90 \text{ rad/s}, \ t = 12 \text{ s}$$

$$\omega_2 = \omega_1 + \alpha t$$

$$90 = \omega_1 + 5 \times 12$$

$$\omega_1 = 30 \text{ rad/s}$$

$$\text{Initial velocity} = 30 \text{ rad/s}$$

$$\theta = \omega_1 t + \frac{1}{2} \alpha t^2$$

$$= 30 \times 12 + \frac{1}{2} \times 5 \times 12^2$$

$$= 360 + 360$$

$$= 720 \text{ rad}$$

Angle turned during 12 s is 720 rad

Example 5.14.

A wheel starting from rest is accelerated at the rate of 5 rad/s^2 for an interval of 10 s. If it is then made to stop in next 5 s by applying brakes, determine,

(i) the maximum velocity attained,

(ii) the total angle turned.

Solution

$$\omega_1 = 0$$

$$\alpha = 5 \text{ rad/s}^2$$

$$t = 10 \text{ s}$$

(i) Maximum velocity

$$\omega_2 = \omega_1 + \alpha t$$

$$= 0 + 5 \times 10 = 50$$

Maximum velocity = 50 rad/s

(ii) Angle turned during 10 s.

$$\theta = \omega_1 t + \frac{1}{2} \alpha \times t^2$$

$$= 0 + \frac{1}{2} \times 5 \times 10^2 = 250 \text{ rad.}$$

During the retardation,

$$\omega_1 = 50 \text{ rad/s}, \ \omega_2 = 0, \text{ and } t = 5 \text{ s}$$

$$\omega_2 = \omega_1 + \alpha t$$

$$0 = 50 + \alpha \times 5$$

$$\alpha = -10 \text{ rad/s}^2$$

$$\theta = \omega_1 t + \frac{1}{2} \alpha t^2 = 50 \times 5 + \frac{1}{2} \times (-10) \times 5^2$$

$$= 125 \text{ rad.}$$

$$\text{Total angle turned} = 250 + 125$$

$$= 375 \text{ rad}$$

Example 5.15.

During the starting phase of computer it is observed that a storage disc which was initially at rest executed 2.5 revolutions in 0.5 seconds. Assuming that the angular acceleration of the motion was uniform

Determine.

(i) the angular acceleration of the disc and

(ii) the velocity of disc at t = 0.5 s.

Solution.

$$\omega_1 = 0$$

$$\theta = 2.5 \text{ revolution}$$

$$= 2.5 \times 2\pi = 5\pi \text{ rad}$$

$$t = 0.5 \text{ s}$$

(i) angular acceleration of disc,

$$\theta = \omega_1 t + \frac{1}{2} \alpha t^2$$

$$5\pi = 0 + \frac{1}{2} \times \alpha \times 0.5^2$$

$$\alpha = 125.66 \text{ rad/s}^2$$

(ii) Velocity of disc

$$\omega_2 = \omega_1 + \alpha t$$

$$= 0 + 125.66 \times 0.5$$

$$= 62.83 \text{ rad/s}$$

Example 5.16.

A passanger car is travelling at 65 kmph on a level road. The distance from the road to the centre of the wheel is 30 cm. Determine,

(i) the magnitude of angular velocity of the wheels

(ii) the magnitude of the constant angular deceleration of the wheels, if the car is brought to rest in 150 m.

Solution.

(i) Magnitude of angular velocity

$$V = 65 \text{ kmph} = 18.06 \text{ m/s}$$

$$r = 30 \text{ cm} = 0.3 \text{ m}$$

$$\text{Angular velocity}, \omega = \frac{V}{r} = \frac{18.06}{0.3} = 60.2 \text{ rad/s}$$

(ii) Angular deceleration

$$\omega_1 = 60.2 \text{ rad/s}$$

$$\omega_2 = 0$$

$$S = 150 \text{ m}$$

Consider the linear motion of car

$$V_2^2 = V_1^2 + 2\,aS$$

$$0 = 18.06^2 + 2 \times a \times 150$$

$$a = -1.09 \text{ m/s}^2$$

$$\text{Linear deceleration} = 1.09 \text{ m/s}^2$$

$$\text{Since } a = r\alpha$$

$$\alpha = \frac{a}{r} = \frac{1.09}{0.3} = 3.63 \text{ rad/s}^2$$

Example 5.17.

The angular acceleration of the rotor of an electric motor is, $\alpha = 30\,t$ rad/s2. The rotor starts at time $t = 0$ s. with no initial velocity. After 2.5 s the rotor has completed 5 revolutions write the equations for angular velocity and angular displacement . Also determine the angular velocity and angular displacement at $t = 5$ s.

Solution.

$$\alpha = 30\,t \ \text{rad/s}^2$$

$$\text{at } t = 0,\ \omega = 0$$

$$\text{at } t = 2.5,\ \theta = 5 \times 2\pi \text{ rad} = 10\,\pi \text{ rad}$$

$$\alpha = \frac{d\omega}{dt} = 30\,t$$

$$d\omega = 30\,t \times dt$$

$$\omega = \frac{30\,t^2}{2} + C_1 = 15\,t^2 + C_1$$

$$0 = 15 \times 0 + C_1$$

$$C_1 = 0$$

$$\omega = 15\,t^2$$

The expression for angular velocity is, $\omega = 15\,t^2$

$$\omega = \frac{d\theta}{dt} = 15\,t^2$$

$$d\theta = 15\,t^2\,dt$$

$$\theta = \frac{15\,t^3}{3} + C_2$$

$$\text{at } t = 2.5,\ \theta = 10\pi \text{ rad}$$

$$10\pi = \frac{15}{3} \times 2.5^3 + C_2$$

$$10\pi = 78.125 + C_2$$

$$C_2 = -46.71$$

$$\therefore \theta = \frac{15t^3}{3} - 46.71$$

The rotor
evolutions
he angular

The expression for angular displacement,

$$\theta = \frac{15t^3}{3} - 46.71$$

Expression for angular velocity is

$$\omega = 15\,t^2$$

Expression for angular displacement is

$$\theta = 5\,t^3 - 46.71$$

$$\text{at } t = 5\text{ s.}$$

$$\text{Angular velocity } \omega = 15 \times 5^2 = 375 \text{ rad/s}$$

$$\text{Angular displacement } \theta = 5 \times 5^3 - 46.71$$

$$= 578.29 \text{ rad}$$

5.3. Equation of motion of rigid body rotating about a fixed axis.

Consider a body of mass m rotating with an angular acceleration α rad/s^2 about a fixed axis O as shown in Fig. 5.9. Consider an elementary mass dm at a radius r and rotating with an angular velocity ω rad/s. The elementary mass is subjected to an acceleration with components a_n and a_t, normal and tangential accelerations , given by

$$a_n = a_y = \omega^2 r \text{ and}$$

$$a_t = r\,\alpha$$

The tangential force acting on the elementary mass

$$= \text{mass} \times \text{ tangential acceleration}$$

$$= dm \times r\alpha$$

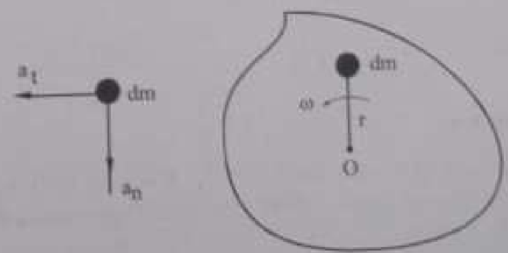

Fig. 5.9

Moment of this force about O,

$$= dm\ r\alpha \times r$$

$$= dm\ r^2\ \alpha$$

The turning moment or torque on the whole body about O,

$$T = \int dm\ r^2 \alpha$$

$= I\ \alpha$, where I is the mass moment of inertia of the body.

$$I = \int dm\ r^2$$

The force due to normal component of acceleration is directed towards the centre of rotation and hence its moments about O is zero

$\therefore$ The turning moment or torque, $T = I\ \alpha$.

It is analogus to Newton's law of rectilinear motion F = ma. Just as linear motion is caused by force, rotation motion is caused by torque.

Kinetic energy due to rotation.

Consider the elementary mass dm in Fig. 5.9.

Kinetic energy of elementry mass $= \frac{1}{2}\ dm\ V^2 = \frac{1}{2}\ dm\ \omega^2 r^2 = \frac{1}{2}\ dm\ r^2\omega^2$

Kinetic energy of the whole body,

$$KE = \int \frac{1}{2}\ dm\ r^2\ \omega^2$$

$$= \frac{1}{2}\ I\ \omega^2$$

Work done in rotation.

Consider a body moving along a circular path. When the body is at A as shown in Fig 5.10, the tangential force $F = m \times a_t$. Work done by this force during the displacement ds in t second is $F \times ds = F \times r\theta$

Work done $= T \times \theta$

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When the body rotates at an angular speed of N rpm, ($\frac{N}{60}$ rps) , the work done/s

$$= T \times \text{angle turned/s}$$

$$= T \times 2\pi \times \frac{N}{60}$$

$$= \frac{2\pi NT}{60} \text{ watts.}$$

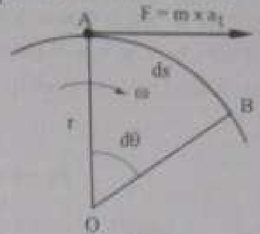

Fig. 5.10

Work energy equation for rotation.

The work energy principle states that the work done by a torque acting on a body during an angular displacement is equal to the change in kinetic energy of the body during the same displacement. Consider a body of mass moment of inertia I moving with an angular velocity ω_1. Let θ be the angular displacement of the body and ω_2 be the final angular velocity. Let T be the resultant torque acting on the body in the direction of angular displacement.

Resultant torque $T = I \times \alpha$, where α is the angular acceleration of the body.

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} \times \frac{d\theta}{dt} = \frac{d\omega}{d\theta} \times \omega$$

$$\alpha = \omega \frac{d\omega}{d\theta}$$

$$T = I \times \alpha$$

$$= I \times \omega \frac{d\omega}{d\theta}$$

$$T \times d\theta = I \; \omega \, d\omega$$

Integrating on both sides,

$$\int_0^\theta T \, d\theta = \int_{\omega_1}^{\omega_2} I \, \omega \, d\omega$$

$$T \times \theta = I \left[\frac{\omega^2}{2}\right]_{\omega_1}^{\omega_2}$$

$$= \frac{1}{2} I (\omega_2^2 - \omega_1^2)$$

$$= \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2$$

Work done = Change in K.E

Example 5.18.

A string 5 m long is wound around the axle of a wheel. The string is pulled with a constant force of 250 N. The wheel rotates at 300 rpm when the string leaves the axle. Find the moment of inertia of the wheel.

Solution:

$$\text{Force} = 250 \text{ N}$$

$$\text{Length of string} = 5 \text{ m}$$

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 10\pi \text{ rad/s}$$

Change in K.E. = work done

$$\left[\frac{1}{2} I\omega^2 - 0\right] = T\theta$$

$$= F \times r \times \frac{5}{r}$$

$$= F \times 5$$

$$= 250 \times 5 = 1250 \text{ Nm}$$

$$r\theta = 5$$

$$\theta = \frac{5}{r}$$

Fig. 5.11

$$\frac{1}{2} \times I \times (10\pi)^2 = 1250$$

$$I = \frac{2500}{(10\pi)^2} = 2.53 \text{ kgm}^2$$

Example 5.19

A flywheel weighing 50 kN and having radius of gyration 1m loses its speed from 400 rpm to 300 rpm in 120 seconds. Calculate

(i) The retarding torque acting on it

(ii) Change in the kinetic energy during the above period.

Solution:

$$N_1 = 400 \text{ rpm}, \; N_2 = 300 \text{ rpm}$$

$$t = 120 \text{ s}, \quad W = 50 \times 10^3 \text{ N}$$

$$k = 1 \text{ m}.$$

(i) Retarding torque

$$T = I\alpha$$

$$= m k^2 \alpha$$

$$\omega_2 = \omega_1 + \alpha t$$

$$\frac{2\pi \times 300}{60} = \frac{2\pi \times 400}{60} + \alpha \times 120$$

$$\alpha = -0.087 \text{ rad/s}^2$$

$$T = I\alpha = m k^2 \alpha$$

$$\therefore T = \frac{50 \times 10^3}{9.81} \times 1^2 \times 0.087$$

$$= 443.43 \text{ N.m}$$

(ii) Change in K.E.

Change in KE = Initial KE – Final KE

$$= \frac{1}{2} I \omega_1^2 - \frac{1}{2} I \omega_2^2 = \frac{1}{2} I (\omega_1^2 - \omega_2^2)$$

$$\omega_1 = \frac{2\pi N_1}{60} = \frac{2\pi}{60} \times 400 = 41.89 \text{ rad/s}$$

$$\omega_2 = \frac{2\pi N_2}{60} = \frac{2\pi}{60} \times 300 = 31.42 \text{ rad/s}$$

$$\text{Change in K.E.} = \frac{1}{2} \times mk^2 (\omega_1^2 - \omega_2^2)$$

$$= \frac{1}{2} \times \frac{50 \times 10^3}{9.81} (41.89^2 - 31.42^2)$$

$$= 1956.05 \text{ kNm}$$

Example 5.20

A right circular disc of weight 1500N and 750 mm diameter is free to rotate about its geometric axis and is constantly accelerated from rest to 300 rpm in 20s. Determine the constant torque required to produce this acceleration.

Solution:

$$\omega_2 = \omega_1 + \alpha t$$

$$\frac{2\pi \times 300}{60} = 0 + \alpha \times 20$$

$$\alpha = \frac{2\pi \times 300}{60 \times 20} = 1.57 \text{ rad/s}^2$$

Moment of inertia of disc, $I = mk^2$

$$= \frac{W}{g}\frac{r^2}{2}$$

$$= \frac{1500}{9.81} \times \frac{\left(\frac{0.75}{2}\right)^2}{2}$$

$$= 10.75 \text{ kg m}^2$$

Torque, $T = I\alpha$

$$= 10.75 \times 1.57$$

$$= 16.88 \text{ Nm}$$

Example 5.21

A shaft of radius r rotates with constant angular speed ω in bearings for which the coefficient of friction is μ. Through what angle θ will it rotate after the driving torque is removed.

Solution:

Frictional force $= \mu R_N = \mu W$.

Torque due to frictional force $= \mu Wr$.

Initial angular velocity $\omega_1 = \omega$

Final angular velocity $\omega_2 = 0$

$$T = \mu Wr$$

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$$\alpha = \frac{\mu W r}{I}$$

$$= \frac{\mu W r}{m\frac{r^2}{2}}$$

$$\alpha = \frac{2\mu g}{r}$$

$$\omega_2^2 = \omega_1^2 - 2\alpha \times \theta$$

$$0 = \omega^2 - 2 \times \alpha\,\theta$$

$$\theta = \frac{\omega^2}{2\alpha}$$

Substituting for α,

$$\theta = \frac{\omega^2}{2 \times \left(\frac{2\mu g}{r}\right)}$$

$$\theta = \frac{\omega^2 r}{4\mu g} \text{ rad}$$

5.4. Rotation under a constant moment.

The equation of a rigid body rotating about a fixed axis is given by $T = I\alpha$.

$$T = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = \frac{T}{I}$$

When the moment T is a constant,

$$\int \frac{d^2\theta}{dt^2} = \frac{T}{I}t + c_1$$, where c_1 is a constant of integration.

$$\frac{d\theta}{dt}=\frac{T}{I}t+c_1$$

Integrating, $\theta=\frac{T}{I}\times\frac{t^2}{2}+c_1t+c_2$, c_2 is some other constant. The values of constants c_1 and c_2 can be obtained by applying the given conditions regarding the angular velocity $\frac{d\theta}{dt}$ and angular displacement θ.

Example 5.22

A solid right circular drum of radius 0.3 m and weight 150 N is free to rotate about its geometric axis as shown in Fig 5.12. Wound around the circumference of the drum is a flexible cord carrying at its free end a weight 45N. If the weight is released from rest, find the time 't' required for it to fall through the height 3m

Solution:

Let P be the tension in the string.

Torque $T = P \times r$

$$T = I\alpha = mk^2 \times \frac{a}{r}$$

$$= m\frac{r^2}{2}\times\frac{a}{r}=\frac{mar}{2}$$

$$Pr=\frac{mar}{2}$$

$$P=\frac{ma}{2}$$

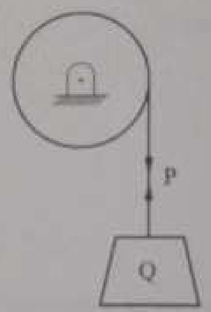

Fig. 5.12

Consider the vertical motion of body Q.

$$Q-P=\frac{Q}{g}\times a$$

$$Q-\frac{ma}{2}=\frac{Q}{g}a$$

$$Q=a\left(\frac{Q}{g}+\frac{m}{2}\right)$$

$$45 = a\left(\frac{45}{9.81} + \frac{150}{2\times 9.81}\right) = 12.23$$

$$a = 3.68\ m/s^2$$

$$h = ut + \frac{1}{2}at^2$$

$$3 = 0 + \frac{1}{2}\times 3.68\times t^2$$

$$t = 1.28\ s$$

Example 5.23

The rotor and the shaft weights 2500 N and the radius of gyration with respect to the axis of rotation is 250 mm. Calculate the acceleration of the falling weight 450 N if the shaft radius is 125 mm.

Solution:

Let P be the tension in the string

Torque due to the weight, $T = P\times r$

$$T = I\alpha$$

$$P\times r = mk^2 \times \frac{a}{r}$$

$$P = mk^2\frac{a}{r^2} = \frac{2500}{9.81}\times\frac{0.25^2}{0.125^2}\times a$$

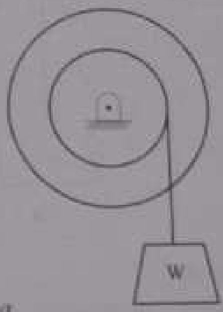

Fig. 5.13

$$P = 1019.37\ a$$

Consider the downward motion of weight W

$$W - P = \frac{W}{g}\times a$$

$$W = P + \frac{W}{g}a$$

$$= 101937\,a + \frac{450}{9.81}\times a$$

$$= 1065.24\ a$$

$$a = \frac{450}{1065.24}$$

$$= 0.42 m/s^2$$

Example 5.24

A pulley of 600 N has a radius of 60 cm. A weight of 400 N is suspended by a string wound round the pulley. The other end of the string is attached to the pulley as shown in Fig 5.14. Determine the acceleration of the weight and the tension in the string.

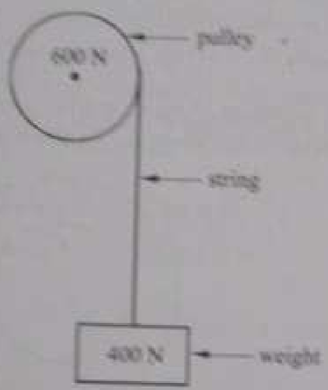

Fig. 5.14

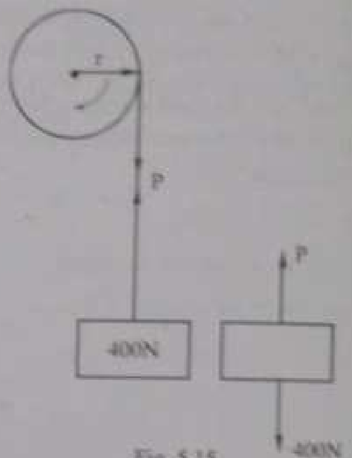

Fig. 5.15

Solution.

Let tension in the string be P

External torque $= T = P \times r$

$$T = I\alpha$$

$$\alpha = \frac{T}{I} = \frac{P \times r}{m k^2}$$

$$= \frac{P \times 0.6}{\frac{600}{9.81} \times \frac{0.6^2}{2}}$$

$$\alpha = 0.0545 P$$

Consider the downward motion of weight 400 N.

Net force = m a

$$400 - P = \frac{400}{9.81} \times a \qquad \begin{bmatrix} a = r\,\alpha \\ = 0.6\alpha \end{bmatrix}$$

$$400 - P = \frac{400}{9.81} \times 0.6 \times \alpha$$

$$400 - P = \frac{400}{9.81} \times 0.6 \times 0.0545\,P$$

$$400 - P = 1.33\,P$$

$$P = 171.67\text{ N}$$

$$\alpha = 0.0545\,P$$

$$= 0.0545 \times 171.67$$

$$= 9.36\text{ rad/s.}$$

$$a = r\,\alpha$$

$$= 0.6 \times 9.36$$

$$= 5.62\text{ m/s}^2$$

Acceleration of weight = 5.62 m/s^2

Tension in the string = 171.67 N.

Example 5.25

A right circular cylinder of mass 500 kg and radius 10 cm is suspended from a string that is wound around its circumference. If the cylinder is allowed to fall freely, find the acceleration of its mass centre and the tension in the string.

Solution.

Let the tension in the string be P, acceleration of mass centre "a" and angular acceleration of cylinder α.

Consider the downward motion of cylinder.

Net force = mass × acceleration

$$m\,g - P = m \times a$$

$$500 \times 9.81 = P + 500\,a$$

$$P + 500\,a = 4905 \text{ ---------- (i)}$$

Torque, $T = P \times r = I\alpha$

$$= \frac{m\,r^2}{2} \times \alpha$$

$$= \frac{m r^2}{2} \times \frac{a}{r}$$

$$T = P \times r = \frac{m a r}{2}$$

$$P = m \frac{a}{2} = 250 a - - - -(ii)$$

Substituting the value of P in eqn (i)

$$250 a + 500 a = 4905$$

$$a = 6.54 \text{ m/s}^2$$

$$P = 250 \times a$$

$$= 250 \times 6.54$$

$$= 1635 \text{ N}$$

Example 5.26

Two blocks of masses 10 kg and 25 kg are attached to the two ends of a flexible rope. The rope passes over a pulley of diameter 500 mm. The mass of the pulley is 7.5 kg and its radius of gyration is 200 mm. Find the acceleration of the masses and the tension on either side of the rope.

Solution:

Consider the downward motion of block B

$$m_B g - T_B = m_B \times a$$

$$25 \times 9.81 - T_B = 25 a \text{ ....(i)}$$

Consider the upward motion of block A

$$T_A - m_A g = m_a a$$

$$T_A - 10 \times 9.81 = 10 \times a \text{ ....(ii)}$$

Consider the rotation of pulley

$$\text{Net torque} = I\alpha$$

$$T_B \times r - T_A \times r = mk^2 \times \frac{a}{r}$$

$$T_B - T_A = 7.5 \times 0.2^2 \times \frac{a}{0.25^2}$$

$$T_B - T_A = 4.8\,a \text{ ....(iii)}$$

From eqn (i)

$$T_B = 25 \times 9.81 - 25\,a$$

From eqn (ii)

$$T_A = 10\,a + 10 \times 9.81$$

$$T_B - T_A = 15 \times 9.81 - 35\,a$$

$$4.8\,a = 15 \times 9.81 - 35\,a$$

$$39.8\,a = 15 \times 9.81$$

$$a = 3.7 \text{ m/s}^2$$

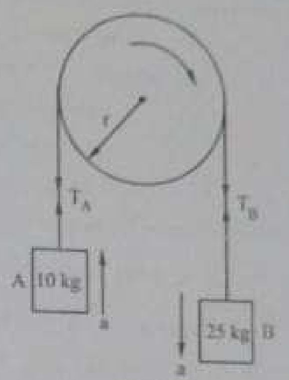

Fig. 5.16

From eqn (i)

$$25 \times 9.81 - T_B = 25 \times 3.7$$

$$T_B = 152.75 \text{ N}$$

$$T_B - T_A = 4.8\,a$$

$$T_A = T_B - 4.8 \times 3.7$$

$$= 152.75 - 4.8 \times 3.7$$

$$= 135 \text{ N}$$

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Example 5.27

The composite pulley shown in Fig 5.17 weighs 800 N and has a radius of gyration of 0.6m. The 200 kg and 400 kg blocks are attached to the pulley by strings as shown. Determine the tension in the string and angular acceleration of the pulley..

Solution.

Let a_1 be the downward acceleration of block of mass 400 kg and a_2 be the upward acceleration of 200 kg block and α. be the angular acceleration of pulley.

$$a_1 = r_1\,\alpha. \text{ and } a_2 = r_2\,\alpha.$$

$$a_1 = 0.5\,\alpha. \text{ and } a_2 = 0.75\,\alpha.$$

$$\frac{a_2}{a_1} = \frac{0.75\,\alpha}{0.5\,\alpha} = 1.5$$

$$a_2 = 1.5\,a_1 \text{ --------------- (i)}$$

Equation for the motion of the two blocks,

$$\text{Net force} = ma$$

$$400 \times g - T_1 = 400\,a_1 \text{ --------------- (ii)}$$

$$T_2 - 200\,g = 200 \times a_2$$

$$T_2 - 200\,g = 200 \times 1.5\,a_1$$

$$T_2 - 200\,g = 300\,a_1 \text{ -------- (iii)}$$

Equation of motion for the pulley.

$$\text{Net Torque} = I\,\alpha$$

$$T_1 \times 0.5 - T_2 \times 0.75 = M\,k^2 \times \frac{a_1}{0.5}$$

$$0.5\,T_1 - 0.75\,T_2 = \frac{800}{9.81} \times 0.6^2 \times \frac{a_1}{0.5}$$

$$0.5\,T_1 - 0.75\,T_2 = 58.72\,a_1 \text{ -------- (iv)}$$

$$T_1 = 1.5\,T_2 + 117.44\,a_1.$$

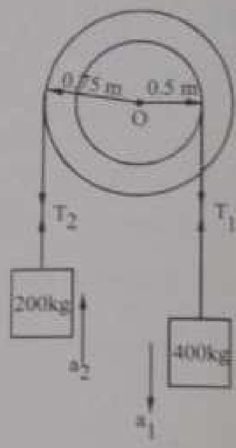


Fig. 5.17

Substituting this value of T_1 in eqn (ii),

$$400\,g - 1.5\,T_2 - 117.44\,a_1 = 400\,a_1$$

$$-T_2 + 266.67\,g = 344.96\,a_1 \text{ -------- (v)}$$

Adding eqn (iii) and eqn (v)

$$66.67\,g = 644.96\,a_1$$

$$a_1 = 1.01 \text{ m/s}^2$$

$$a_2 = 1.5a_1 = 1.52 \text{ m/s}^2$$

$$\alpha = \frac{a_1}{0.5} = 2.02 \text{ rad/s}^2$$

$$T_2 = 200\,g + 300\,a_1 = 200 \times 9.81 + 300 \times 1.01$$

$$T_2 = 2265 \text{ N}$$

$$T_1 = 1.5\,T_2 + 117.44\,a_1$$

$$= 1.5 \times 2202 + 117.44 \times 1.01 = 3516.11 \text{ N}$$

5.5. Plane motion of rigid body

A motion is said to be a rotation motion when the particles of a body move along concentric circles. In translation all the particles forming the body move along parallel paths. If these paths are straight lines, the motion is said to be a rectilinear translation, if the paths are

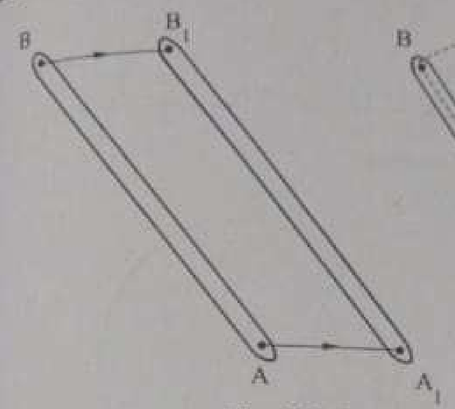


Fig. 5.18. Translation

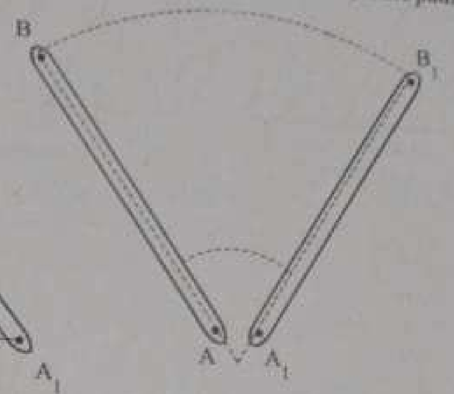


Fig. 5.19. Rotation

curved lines, the motion is a curvilinear translation. Any plane motion which is neither a translation nor a rotation is referred to as a general plane motion . Rolling, without slipping, is an example of plane motion.For the analysis of general plane motion, it is convenient to split the motion into translation and pure motion of rotation. The analysis for these two cases is carried out separately and then combined to get the final motion of rotation and translation, as shown in figures 5.20 and 5.21.

Wheels rolling without slipping.

When a wheel of radius r rolls without slipping along a horizontal straight path, as shown in Fig 5.22. The linear distance AB will be equal to the angular distance AB.

Linear distance AB = Angular distance AB

$$S_c = r\theta.$$

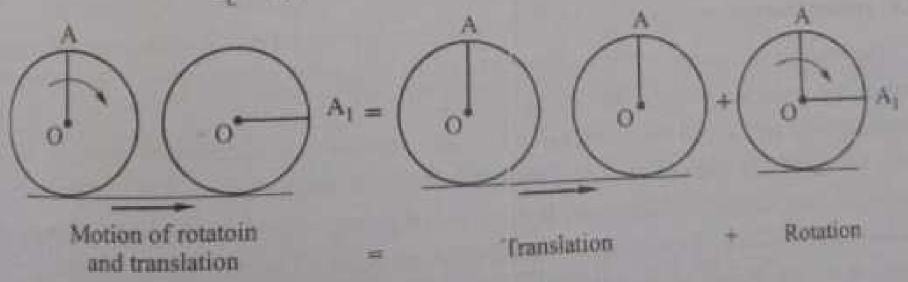


Fig. 5.20

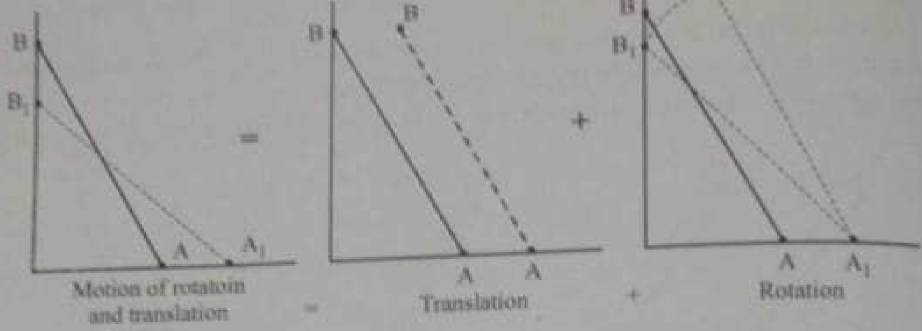
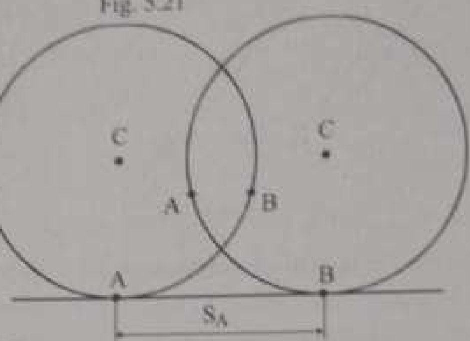


Fig. 5.21



Velocity of C, $V_C = \frac{dS_C}{dt} = r\frac{d\theta}{dt} = r\omega$

Fig. 5.22

Acceleration of C, $a_C = \frac{dV_C}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$

Thus when a wheel of radius r, rotates with angular velocity ω, without slipping, then the displacement, velocity and acceleration of the geometric centre of the wheel are given by

$$S_C = r\theta$$

$$V_C = r\omega$$

$$a_C = r\alpha$$

5.6. Instantaneous centre of rotation.

The motion of rotation and translation of a body, at a given instant, can be considered as that of pure rotation of the body about a point. This point about which the body can be assumed to be rotating at the given instant is called instantaneous centre of rotation. Since the velocity of this point at the given instant is zero, this point is also called instantaneous centre of zero velocity. This point is not a fixed point, and when the body changes its position, the position of instantaneous centre also changes. The locus of the instantaneous centre as the body goes on changing its position is called centrode.

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Consider the plane motion of a body AB from A_1B_1 to A_2B_2. This plane motion can be considered as pure rotation of body AB about the point I, the instantaneous centre of rota

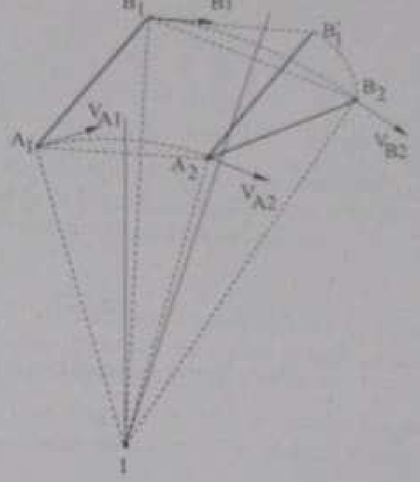

Fig. 5.23

tion, which is the point of intersection of perpendicular bisectors of A_1A_2 and B_1B_2. Since I is on the perpendicular bisector of A_1A_2, $IA_1 = IA_2$. Similarly $IB_1 = IB_2$. Since $AB = A_1B_1 = A_2B_2$, the triangles IA_1B_1 and IA_2B_2 are congruent. Hence the motion of AB from A_1B_1 to A_2B_2 is pure rotation of triangle IA_1B_1 about I. Hence A_1B_1, at the given instant rotates about I. Since A_1B_1 rotates about I, the magnitude of velocity of A_1 is $\omega \times IA_1$ and that of B_1 is $\omega \times IB_1$.

The direction of velocity of points A_1 and B_1 are perpendicular to IA_1 and IB_1 respectively. Thus the properties of instantaneous centre are,

(i) the magnitude of velocity of any point on a body is proportional to its distance from the instantaneous centre and is equal to the angular velocity times the distance.

(ii) The direction of velocity of any point on a body is perpendicular to the line joining that point and the instantaneous centre.

The above properties are used to locate the instantaneous centre of a body.

Case (i)

When the direction of velocities of any two points on the body are known.

In this case the point of intersection of lines drawn perpendicular to the direction of velocities will be the instantaneous centre. Refer Fig. 5.24.

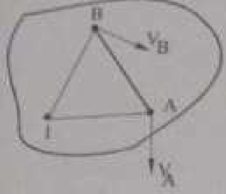

Fig. 5.24

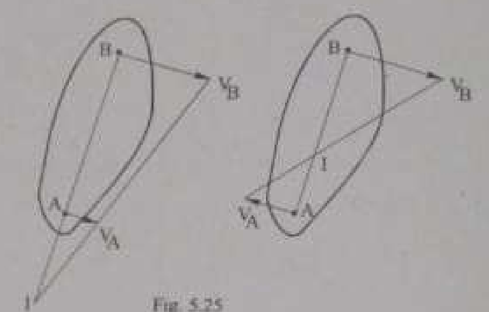

Fig. 5.25

Case (ii)

When the direction of velocities are parallel and magnitudes are unequal.

In this case the point of intersection of the line joining the tip of velocity vectors and either the line joining the points or extension of the line joining the points will be the instantaneous centre. Refer Fig. 5.25.

Example 5.28.

A cylinder of radius 1 m rolls without slipping along a horizontal plane AB. Its centre has a uniform velocity of 20 m/s. Find the velocities of the points D and E on the circumference of the cylinder as shown in Fig. 5.26.

Solution

Since the linear velocity of point O is zero, it is the instantaneous centre.

Velocity of point C.

$V_C \propto OC$

$V_C = \omega \times OC$

$20 = \omega \times 1$

$\omega = 20$ rad/s

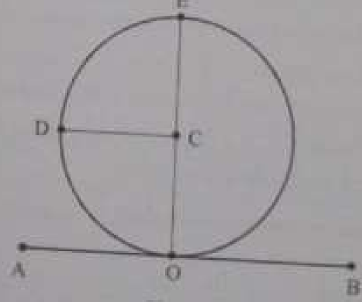

Fig. 5.26

Velocity of point E

$V_E \propto OE$

$V_E = \omega \times OE$

$= 20 \times 2$

Velocity of point D

Example 5.29.

A bar AB rests at
in Fig 5.27. The ba
constant velocity of

Solution.

The direction of
BA. The instantane
perpendicular to the

AC sin 60 = 3 m

$AC = \frac{3}{\sin 60}$

AI cos 30 = AC = 3

$AI = \frac{3.46}{\cos 30}$

$V_A = \omega_{AB}$

$\omega_{AB} = \frac{V_A}{AI}$

Example 5.30.

A cylindrical roll
as shown in Fig. 5.2
the angular velocity

(i) the velocities

(ii) the direction

$= 40 \text{ m/s}$

Velocity of point D

$$V_D = \omega \times OD$$

$$OD = \sqrt{OC^2 + CD^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$V_D = 20 \times \sqrt{2} = 28.28 \text{ m/s}$$

Example 5.29.

A bar AB rests at the edge of a wall of height 3 m, at some point C on the bar as shown in Fig 5.27. The bar makes an angle of 60° with horizontal. If the end A moves with a constant velocity of 6 m/s, find the angular velocity of the bar.

Solution

The direction of velocity of A is horizontal and direction of velocity of C is along the bar BA. The instantaneous centre of bar AB is the point of intersection of the lines drawn perpendicular to the direction of velocities V_A and V_C.

$AC \sin 60 = 3 \text{ m}$

$$AC = \frac{3}{\sin 60} = 3.46 \text{ m}$$

$AI \cos 30 = AC = 3.46$

$$AI = \frac{3.46}{\cos 30} = 4 \text{ m}$$

$$V_A = \omega_{AB} \times AI$$

$$\omega_{AB} = \frac{V_A}{AI} = \frac{6}{4} = 1.5 \text{ rad/s}$$

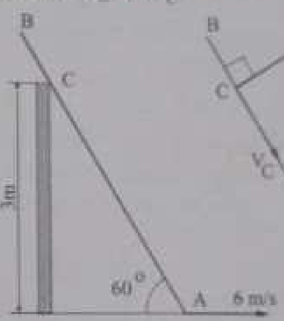

Fig. 5.27

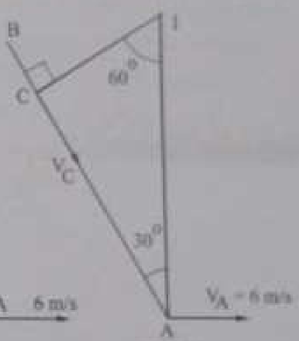

Fig. 5.28

Example 5.30.

A cylindrical roller is in contact at its top and bottom, with two conveyor belts AB and DE as shown in Fig. 5.29. If the belts run at uniform speeds of 3 m/s and 2 m/s respectively, find the angular velocity of the roller, when,

(i) the velocities are in the same direction and

(ii) the direction of velocities are opposite. The diameter of the roller is 40 cm.

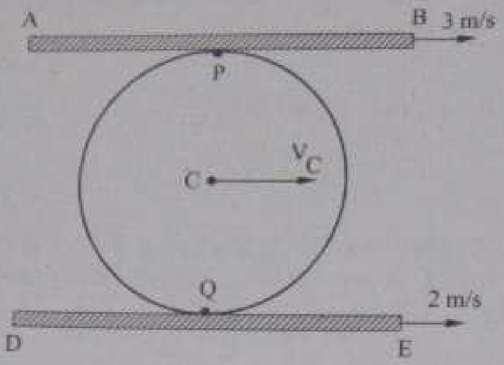

Fig. 5.29

Solution

Case (i) when the velocities are in the same direction.

Let the position of instantaneous centre be at a distance x below Q,

Velocity of P, $3 = \omega\ (0.4 + x)$

Velocity of Q, $2 = \omega x$

Fig. 5.30

$$\frac{3}{2} = \frac{0.4 + x}{x}$$

$$3x = 0.8 + 2x$$

$$x = 0.8$$

$$2 = \omega x$$

$$\omega = \frac{2}{0.8} = 2.5 \text{ rad/s}$$

Case (ii) when the velocities are in opposite directions.

Let the instantaneous centre be at a distance x above the point Q

$$\omega \times IP = \omega\,(0.4 - x)$$

$$2 = \omega \times IQ = \omega x$$

$$\frac{3}{2} = \frac{0.4 - x}{x}$$

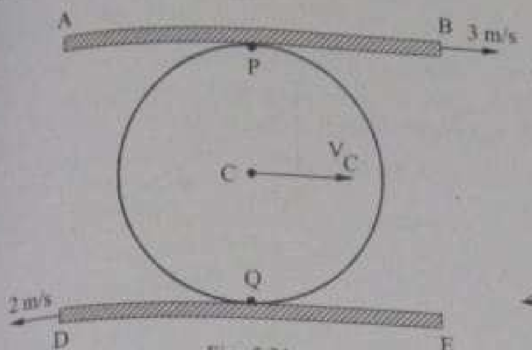

Fig. 5.31

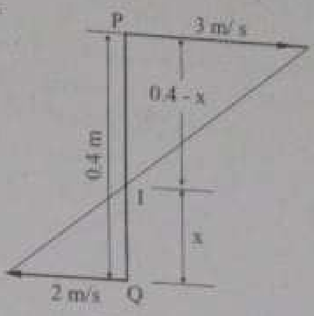

Fig. 5.32

$$3x = 0.8 - 2x$$

$$5x = 0.8 \text{ m}$$

$$x = 0.16 \text{ m}$$

$$2 = \omega \times x$$

$$\omega = \frac{2}{0.16} = 12.5 \text{ rad/s}$$

Example 5.31. [KTU May 2019]

A bar PQ of length 1 m has its ends P and Q constrained to move horizontally and vertically as shown in Fig. 5.33. The end P moves with constant velocity of 5 m/s horizontally. When the bar makes an angle of 30° with the horizontal

find,

(i) the angular velocity of the bar

(ii) the velocity of the end Q and

(iii) the velocity of the mid point M

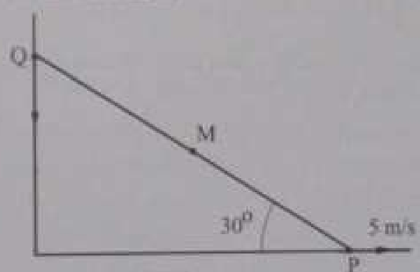

Fig. 5.33

Solution

Since P is moving horizontally and Q is moving vertically, the instantaneous centre is the point of intersection of lines drawn perpendicular to the velocities of P and Q

$$V_P = \omega \times IP$$

$$V_Q = \omega \times IQ$$

$$\frac{V_Q}{V_P} = \frac{IQ}{IP} = \tan 60$$

Velocity of end Q, $V_Q = V_P \tan 60$

$= 5 \times \tan 60$

$= 8.66$ m/s

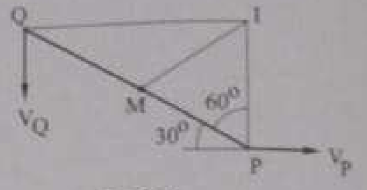

Fig. 5.34

$V_P = \omega \times IP$

$= \omega \times PQ \cos 60$

$5 = \omega \times 1 \times 0.5$

Angular velocity of bar $\omega = 10$ rad/s

Velocity of mid point M,

$V_M = \omega \times IM$

$IM = \sqrt{0.5^2 + 0.5^2 - 2 \times 0.5 \times 0.5 \cos 60}$

$= 0.5$ m

$V_M = \omega \times IM = 10 \times 0.5 = 5$ m/s

Example 5.32.

A compound wheel rolls without slipping as shown in Fig.5.35. The velocity of centre C is 2 m/s. Find the velocities of the points A, and B

Solution.

Since the smaller wheel rolls without slipping, velocity point O is zero. O is the instantaneous centre.

$V_C = \omega \times OC$

$2 = \omega \times 0.1$

$\omega = 20$ rad/s

$V_A = \omega \times OA$

$= 20 \times 0.3$

$= 6$ m/s

$V_B = \omega \times OB$

$= 20 \times 0.1$

$= 2$ m/s

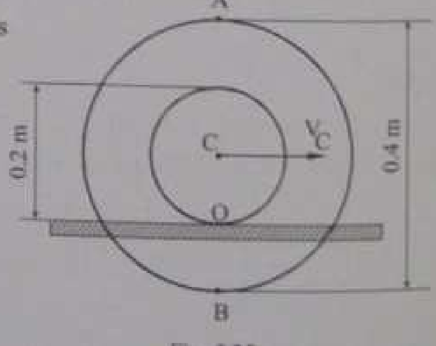

Fig. 5.35

Example 5.33.

The end A of a bar AB moves with a constant velocity of 4 m/s along the horizontal surface. The bar always remains in contact with a semicircular disc of radius 6 cm which is resting on the horizontal surface as shown in Fig.5.36. Find the angular velocity of the bar at an instant when the bar makes an angle of 30° with the horizontal.

Solution.

$$V_A = \omega_{AB} \times AI$$

From the triangle OAC

$$OC = OA \cos 60$$

$$OA = \frac{OC}{\cos 60} = 12\,\text{cm}$$

From the triangle OAI

$$\frac{IA}{OA} = \tan 60$$

$$IA = OA \tan 60$$

$$= 12 \tan 60 = 20.78$$

$$V_A = \omega_{AB} \times AI$$

$$4 = \omega_{AB} \times 20.78$$

$$\omega_{AB} = 0.19\ \text{rad/s}$$

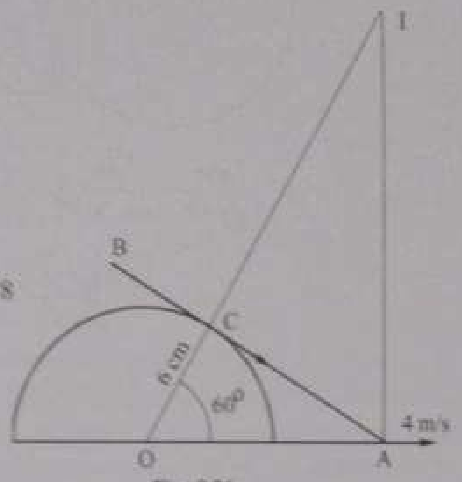

Fig. 5.36

Example 5.34. [KTU Jan 2016, Jun 2016, August 2016, Jan 2017, May 2017]

In the reciprocating engine mechanism shown in Fig. 5.37 The crank OA rotates at a uniform speed of 300 rpm. The lengths of crank and connecting rod are 12 cm and 50 cm respectively. Find (i) the angular velocity of the connecting rod AB and (ii) the velocity of piston when the crank makes an angle of 30° with horizontal.

Solution

The direction of velocity of point A is perpendicular to OA. The direction of velocity of point B is horizontal. Therefore the point of intersection of the line OA produced and the vertical from B is the intantaneous centre I

$$AB = OA \sin 30 = AB \sin \phi$$

$$\sin \phi = \frac{OA}{AB} \sin 30$$

$$= \frac{12}{50} \sin 30 = 0.12$$

$$\phi = 6.89^{0}$$

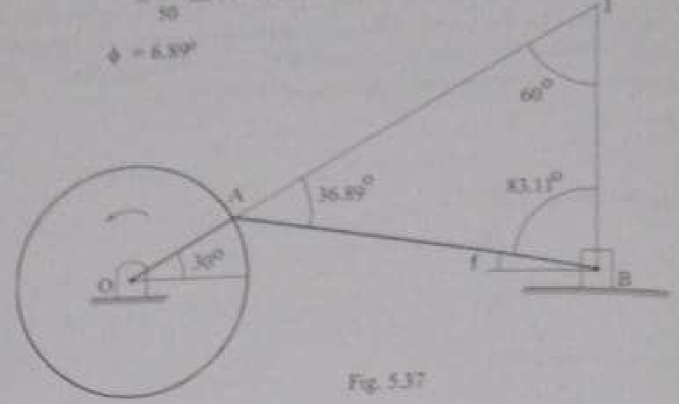

Fig. 5.37

$$\frac{AB}{\sin 60} = \frac{BI}{\sin 36.89} = \frac{AI}{\sin 83.11}$$

$$BI = \frac{AB \sin 36.89}{\sin 60} = \frac{50 \sin 36.89}{\sin 60} = 34.66 \text{ cm}$$

$$AI = \frac{AB \sin 83.11}{\sin 60} = \frac{50 \sin 83.11}{\sin 60} = 57.32 \text{ cm}$$

$$\omega_{OA} = \frac{2\pi N}{60} = \frac{2\pi \times 300}{60} = 31.4 \text{ rad/s}$$

$$V_A = \omega_{AB} \times AI \qquad V_A = \omega_{OA} \times OA = 31.4 \times 0.12 = 3.77 \text{m/s}$$

Angular velocity of connecting to rod,

$$\omega_{AB} = \frac{3.77}{0.5732} = 6.58 \text{ rad/s}$$

Velocity of piston, $V_B = \omega \times BI = 6.58 \times 0.3466 = 2.28$ m/s

5.7. Simple harmonic motion.

Simple harmonic motion (SHM) is a periodic motion. Any motion which repeats after equal interval of time is called a periodic motion. For a periodic motion to be simple harmonic, it should satisfy two general conditions.

(i) The acceleration of the body or particle performing periodic motion should be proportional to the distance of the body or particle from a fixed point called centre of simple harmonic motion (mean position of the body or particle.)

(ii) The acceleration of the body or particle should always be directed towards the mean position.

Consider a particle moving along the circumference of a circle of radius r with a uniform angular velocity ω radians per second. Let P be the position of the particle after t seconds from the start of motion from the position A as shown in Fig. 5.38.

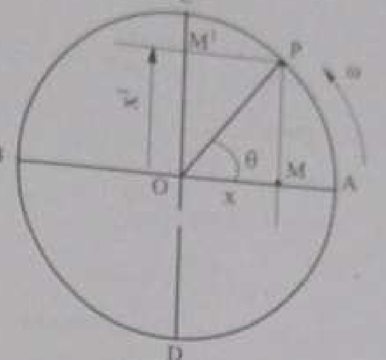

Fig. 5.38

M is the projection of particle on the horizontal diameter AB. When the particle moves along the arc ACB, the point M moves from A to B. Similarly when the particle moves along the arc BDA, the point M moves from B to A. It can be proved that the acceleration of point M is proportional to its distance from O, and is directed towards O. Such a motion is called simple harmonic motion and hence the projection of particle on the horizontal diameter executes SHM. O is the mean position and A and B are the extreme positions. Similarly the projection of particle on the vertical diameter CD also executes SHM with C and D as extreme positions.

Terms used with SHM

The following terms are generally used with SHM.

1. **Amplitude.** It is the distance between extreme and mean position of the particle executing SHM. It is the maximum displacement of the particle from the mean position.

2. **Oscillation.** It is the motion of projection of particle along the diameter. Referring figure 16.2, one oscillation is said to be completed when the point M moves from A to B and then from B to A.

3. **Period.** It is the time for one oscillation. It is denoted by t_p.

From Fig. 5.38,

$\theta = \omega t$. For one oscillation, $t = t_p$ and $\theta = 2\pi$

$$2\pi = \omega t_p$$

$$t_p = \frac{2\pi}{\omega}$$

The displacement of M from the mean position, $OM = x = OP \cos\theta$

$$x = r \cos \omega t$$

Velocity of M, $v = \frac{dx}{dt} = -r\omega \sin \omega t$. The negative sign is because when the time t increases, the displacement x decreases.

The magnitude of velocity, $v = r\omega \sin \omega t$

$$= r\omega \sin\theta$$

$$= r\omega \frac{PM}{OP}$$

$$= r\omega \frac{\sqrt{r^2 - x^2}}{r}$$

$$= \omega\sqrt{r^2 - x^2}$$

Acceleration of M, $a = \frac{dv}{dt} = \frac{d}{dt}(r\omega \sin \omega t)$

$$= -r\omega^2 \cos \omega t$$

$$= -\omega^2 r \cos \omega t$$

$$a = -\omega^2 x.$$

Let M^1 be the projection of particle on the vertical diameter CD.

The displacement of M^1 from the mean position, $x = OP \sin\theta$

$$= r \sin \omega t$$

Velocity of M^1, $v = \frac{dx}{dt} = \frac{d}{dt}(r \sin \omega t)$

$$= r\omega \cos \omega t$$

me t

Acceleration of M', $a = \frac{dv}{dt}$

$$= \frac{d}{dt}(r\,\omega \cos \omega t)$$

$$= -\omega^2 r \sin \omega t$$

$$a = -\omega^2 x$$

Again, $v = r\,\omega \cos \omega t$

$$= r\,\omega \cos \theta$$

$$= r\,\omega \frac{OM}{OP}$$

$$= r\,\omega \frac{\sqrt{r^2 - x^2}}{r}$$

$$v = \omega \sqrt{r^2 - x^2}$$

It should be noted that the expression for displacement $x = r \sin \omega t$ is to be used when the time of motion is measured from the mean position. ie., in the case when at $t = 0$, the particle executing SHM is at the mean position. The expression $x = r \cos \omega t$ should be used when the time of motion is measured from the extreme position. ie., in the case when at $t = 0$, the particle executing SHM is at the extreme position. In both cases the expression for velocity is, $\omega \sqrt{r^2 - x^2}$ and acceleration , $a = -\omega^2 x$.

The maximum velocity is at $x = 0$. ie., at the mean position.

$$v_{max} = \omega \sqrt{r^2 - 0} = r\,\omega.$$

The maximum acceleration is at $x = r$. ie., at the extreme position.

$$a_{max} = -\omega^2 r.$$

At the extreme position, the velocity is zero and at the mean position the acceleration is zero.

Example 5.35 [KTU June 2016]

A body, moving with simple harmonic motion, has an amptitude of 1 m and period of oscillation is 2 seconds. Find the velocity and acceleration of the body at $t = 0.4$ second, when time is measured from (i) the mean position and (ii) the extreme position.

Solution.

Case (i) when time is measured from the mean position.

$$r = 1\text{ m}$$
$$t_p = 2\text{ s}$$
$$t = 0.4\text{ s}$$
$$\omega = \frac{2\pi}{t_p} = \frac{2\pi}{2} = \pi\text{ rad/s}$$
$$\omega t = \pi \times 0.4\text{ rad}$$
$$= (\frac{180}{\pi} \times 0.4\pi)^0$$
$$= 72^0$$
$$x = r \sin \omega t$$
$$x = 1 \sin 72$$
$$= 0.95\text{ m.}$$

Velocity $v = \omega\sqrt{r^2 - x^2} = \pi \times \sqrt{1^2 - 0.95^2}$

$$= 0.98\text{ m/s}$$

Acceleration $a = \omega^2 x$

$$= \pi^2 \times 0.95$$
$$= 9.38\text{ m/s}^2$$

Case (ii) when time is measured from the extreme position.

$$x = r \cos \omega t = 1 \times \cos 72$$
$$= 0.31\text{ m}$$
$$v = \omega\sqrt{r^2 - x^2}$$
$$= \pi\sqrt{1^2 - 0.31^2}$$
$$= 2.99\text{ m/s}$$

Acceleration $a = \omega^2 x$

$$= \pi^2 \times 0.31$$
$$= 3.06\text{ m/s}^2$$

Example 5.36

A body moving with simple harmonic motion has velocities of 10m/s and 4 m/s at 2 and 4 m distance from the mean position. Find the amplitude and time period of the body.

Solution.

$$\text{At } x = 2\text{ m},\ v = 10\text{ m/s}$$

$$\text{At } x = 4\text{ m},\ v = 4\text{ m/s.}$$

$$\text{Velocity, } v = \omega\sqrt{r^2 - x^2}$$

$$10 = \omega\sqrt{r^2 - 2^2}$$

$$4 = \omega\sqrt{r^2 - 4^2}$$

$$\frac{10}{4} = \frac{\sqrt{r^2 - 4}}{\sqrt{r^2 - 16}}$$

$$6.25 = \frac{r^2 - 4}{r^2 - 16}$$

$$6.25\,r^2 - 100 = r^2 - 4$$

$$5.25\,r^2 = 96$$

$$r = 4.28\text{ m}$$

$$10 = \omega\sqrt{r^2 - 2^2}$$

$$10 = \omega\sqrt{4.28^2 - 4}$$

$$= 3.78\,\omega$$

$$\omega = 2.64\text{ rad/s}$$

$$\text{Time period } t_p = \frac{2\pi}{\omega} = \frac{2\pi}{2.64} = 2.38\text{ s}$$

Example 5.37

The piston of an IC engine moves with simple harmonic motion. The crank rotates at 420 rpm and the stroke length is 40 cm. Find the velocity and acceleration of the piston when it is at a distance of 10 cm from the mean position.

Solution.

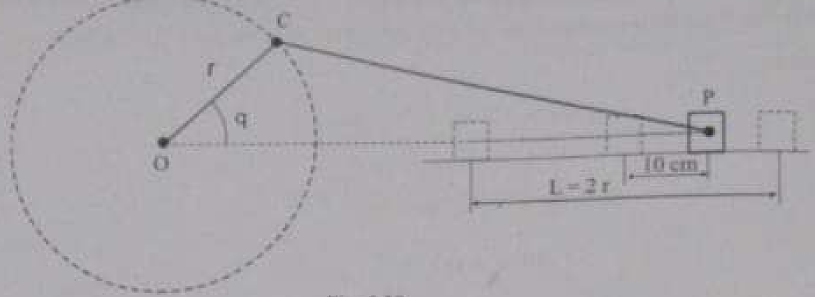

Fig. 5.39

Speed of crank = 420 rpm.

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 420}{60} = 43.98 \text{ rad/s}$$

Stroke length L = 2 × crank radius

$$\text{crank radius, } r = \frac{L}{2} = \frac{40}{2} = 20 \text{ cm}$$

$$= 0.2 \text{ m}$$

$$x = 10 \text{ cm} = 0.1 \text{ m}$$

$$\text{Velocity } v = \omega\sqrt{r^2 - x^2}$$

$$= 43.98\sqrt{0.2^2 - 0.1^2}$$

$$= 7.62 \text{ m/s.}$$

$$\text{Acceleration of the piston } a = \omega^2 r$$

$$= 43.98^2 \times 0.1$$

$$= 193.42 \text{ m/s}^2$$

Example 5.38

A particle moving with SHM has an amplitude of 4.5 m and period of oscillation is 3.5 second. Find the time required by the particle to pass two points which are at a distance of 3.5 m and 1.5 m from the centre and on the same side of mean position.

Solution

Amplitude, $r = 4.5$ m

Time period $t_p = 3.5$ s.

$$\omega = \frac{2\pi}{t_p} = \frac{2\pi}{3.5} = 1.8 \text{ rad/s}$$

Let x_1 and x_2 be the distance of the first and second points from the mean position.

$$x = r \cos \omega t$$

$$x_1 = r \cos \omega t_1$$

$$3.5 = 4.5 \cos (1.8 \times t_1 \times \frac{180}{\pi})^{\circ}$$

$$t_1 = 0.38 \text{ s}$$

$$x_2 = r \cos \omega t_2$$

$$(1.5 = 4.5 \cos (1.8 \times t_2 \times \frac{180}{\pi})$$

$$t_2 = 0.68 \text{ s}$$

Time required to pass the two points,

$$t = t_2 - t_1$$

$$= 0.68 - 0.38$$

$$= 0.3 \text{ s}$$

Example 5.39

A body is vibrating with simple harmonic motion of amplitude 150 mm and frequency 3 cps. Calculate the maximum velocity and acceleration of the body.

Solution.

$$r = 150 \text{ mm} = 0.15 \text{ m}$$

$$f = 3 \text{ cps}$$

$$\omega = 2\pi f$$

$$= 2\pi \times 3 = 6\pi \text{ rad/s}$$

Maximum velocity, $v_{max} = r\omega$

$$= 0.15 \times 6\pi = 2.83 \text{ m/s}$$

Maximum acceleration, $a_{max} = \omega^2 r$

$$= (6\pi)^2 \times 0.15$$

$$= 53.3 \text{ m/s}^2$$

Example 5.40

A prismatic bar AB of weight W is resting on two rough rollers rotating with equal angular velocity in opposite directions as shown in Fig. 5.40. If the bar is so placed on the rollers that its centre of gravity is displaced from the middle plane by a distance x and then released, show that the bar executes simple harmonic motion. Find the period if the co efficient of friction is μ and the distance between the axes of the rollers is $2l$.

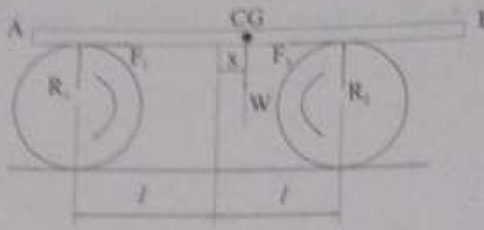

Fig. 5.40

Solution,

Let R_1 and R_2 be the vertical reactions on the bar at the left and right side. F_1 and F_2 are the frictional forces at the surface of the bar. $F_1 = \mu R_1$ and $F_2 = \mu R_2$.
Taking moments about the point of application of reaction R_2,

$$R_1 \times 2l - W(l - x) = 0$$

$$R_1 = \frac{W}{2l}(l - x).$$

Taking moments about the point of application of reaction

$$R_1, R_2 \times 2l - W(l + x) = 0$$

$$R_2 = \frac{W}{2l}(l + x).$$

The net force in the horizontal direction that is in the direction of motion of the bar is $F_1 - F_2$.
$F_1 - F_2$ = mass x acceleration of bar

$$\mu R_1 - \mu R_2 = \frac{W}{g} a$$

$$\mu\frac{W}{2l}(l-x)-\mu\frac{W}{2l}(l+x)=\frac{W}{g}\times a$$

$$\mu\frac{W}{2l}[l-x-(l+x)]=\frac{W}{g}a$$

$$\mu\frac{W}{2l}(-2x)=\frac{W}{g}a$$

$$-\frac{\mu x}{l}=\frac{a}{g}$$

$$a=\frac{-\mu g}{l}x$$

$$\frac{d^2x}{dt^2}=-\frac{\mu g}{l}x.$$

Comparing this equation with equation of S H M, ,

$$\frac{d^2x}{dt^2}=-\omega^2 x.$$

$$\omega^2=\frac{\mu g}{l}$$

$$\omega=\sqrt{\frac{\mu g}{l}}$$

$$\text{Time period, } t_P=\frac{2\pi}{\omega}=2\pi\times\sqrt{\frac{l}{\mu g}}$$

Example 5.41.

A particle is executing simple harmonic motion. Starting from rest it travels a distance a in the first second and a distance b in the next second in the same direction. Show that the amplitude is $\frac{2a^2}{3a-b}$.

Solution

At $t = 1s$, $x = r - a$ and

at $t = 2s$, $x = r - (a+b)$.

The displacement from the mean position $x = r\cos\omega t$

$$(r-a) = r\cos\omega$$

$$\cos\omega = \frac{(r-a)}{r}$$

$$r-(a+b) = r\cos 2\omega$$

$$= r(2\cos^2\omega - 1)$$

$$r-a-b = r\left[2\frac{(r-a)^2}{r^2} - 1\right]$$

$$= 2\frac{(r-a)^2}{r} - r$$

$$= \frac{2(r-a)^2 - r^2}{r}$$

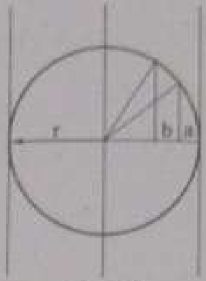

Fig. 5.41

$$r^2 - ar - br = 2(r^2 + a^2 - 2ar) - r^2$$

$$= 2r^2 + 2a^2 - 4ar - r^2$$

$$= r^2 + 2a^2 - 4ar$$

$$4ar - ar - br = 2a^2$$

$$3ar - br = 2a^2$$

$$r = \frac{2a^2}{3a-b}$$

5.8. Free vibration

Vibration of mechanical system generally results when a system is displaced from its position of stable equilibrium. The system tends to return to its equilibrium position due to the action of restoring force. The restoring force may be an elastic force as in the case of mass attached to the end of a spring or gravitational force as in the case of simple pendulum. If a disturbing force is applied just to start the motion and is then removed from the system leaving it to vibrate by itself, the system is said to undergo free vibration. If on the other hand the disturbing force acts at periodic intervals on the system, the system is said to undergo forced vibration. In this book our discussion is limited to free vibrations.

5.9. Degree of freedom.

The number of degrees of freedom of a physical system is the number of independent co ordinates required to define the configuration of the system. A rigid body in space has six degrees of freedom, three co ordinates (x, y and z) to define rectilinear position and three parameters to define the angular positions. The constraints to the motion reduce the degee of freedom of the system. The three systems shown in Fig. 5.42 are single degree of freedom systems. In Fig.5.42(a), when the mass is constrained to move in the vertical

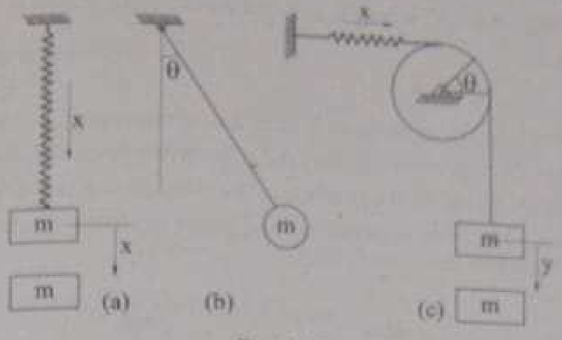

Fig. 5.42

direction only, the displacement of the mass will be same as that of the spring. In Fig.5.42(b) the only one co ordinate required to specify the position of the simple pendulum is θ. In Fig. 5.42(c) the values of x and θ are functions of y and hence the independant co-ordinate required to specify the configuration of system is one.

Fig. 5.43 shows two systems having two degrees of freedom each.

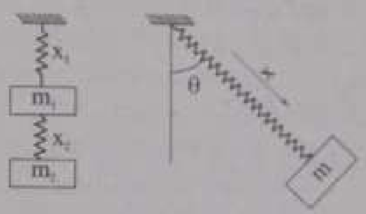

Fig. 5.43

When the spring-mass systems shown in Fig. 5.43 are constrained to move in a vertical plane, two co-ordinates are required to specify the configuration of the system. Hence the degrees of freedom of the systems are two.

5.10. Undamped free vibration of spring mass system

A mass attached to a spring and subjected to a force along the axis of the spring is called a spring mass system. When a spring is compressed or elongated the spring opposes the applied force. This opposing force is called spring force which is proportional to the deformation of the spring. This spring force is proportional to the displacement of the spring. $F \propto x$. Spring force, $F = \text{a constant} \times x$. $F = k\,x$, where k is the stiffness of the spring.

Stiffness $k = \frac{F}{x}$. The unit of stiffness is N/m. Consider a spring of stiffness k. When a body of mass m is attached to one end of this spring, with the other end fixed, the spring elongates.

Let this elongation be δ. This δ is called static deflection of the spring. In the equilibrium position net force on the body is zero.

$mg - k\delta = 0$ where $k\delta$ is the spring force which is opposite to the direction of displacement.

$$mg = k\delta.$$

When the body is displaced by an amount X from the equilibrium position by an external force and if the external force is removed then the body will vibrate between two extreme positions with amplitude X. Consider the position of the body when it is at a distance x below the equilibrium position. The body is in motion and hence net force is equal to mass times acceleration.

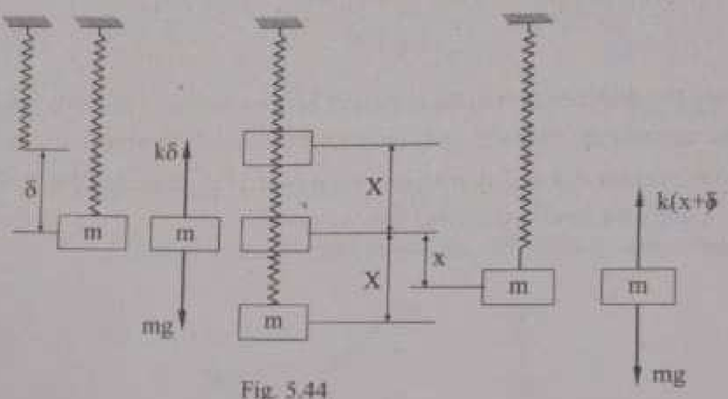

Fig. 5.44

ie., $mg - k(x + \delta) = m \times a$

$$mg - kx - k\delta = m\frac{d^2x}{dt^2}$$

Since $mg = k\delta$,

$$-kx = m\frac{d^2x}{dt^2}$$

$\therefore \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$. This is the equation of motion of free vibration of a body.

$\frac{d^2x}{dt^2} = -\frac{k}{m}x$. Comparing this equation with the equation of simple harmonic motion, $\frac{d^2x}{dt^2} = -\omega^2 x$

$$\omega^2 = \frac{k}{m}$$

$\omega = \sqrt{\frac{k}{m}}$ Since the system vibrates freely, this frequency is called natural frequency and is denoted by ω_n

$\omega_n = \sqrt{\frac{k}{m}}$ Since $\omega = 2\pi f$, $f = \frac{\omega}{2\pi}$

$f_n = \frac{1}{2\pi} \times \sqrt{\frac{k}{m}}$. This is the expression for the natural frequency of spring mass system.

In a spring mass model the springs can be attached to the mass either in series or in parallel. A number of springs having stiffness k_1, k_2, k_3 etc can be replaced by a single spring of stiffness k_e. The stiffness of this single spring, k_e, is called equivalent stiffness. The expression for k_e depends on the arrangement of springs. Fig 5.45 shows three springs of stiffness k_1, k_2, and k_3 arranged in series. Let δ_1, δ_2 and δ_3 be the elongation of each spring due to body of mass m. Static deflection of mass, $\delta = \delta_1 + \delta_2 + \delta_3$. Let δ_e be the static deflection of the same mass when it is attached to the single spring of stiffness k_e. For this single spring to be equivalent to the three springs in series, the required condition is,

$$\delta_e = \delta = \delta_1 + \delta_2 + \delta_3$$

$$mg = k\delta$$

$$\delta = \frac{mg}{k}$$

$$\therefore \frac{mg}{k_e} = \frac{mg}{k_1} + \frac{mg}{k_2} + \frac{mg}{k_3}$$

$$\therefore \frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

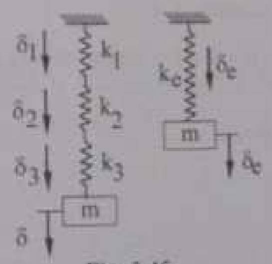

Fig. 5.45

Fig. 5.46 shows three springs arranged in parallel. The required condition for the single spring to be equivalent to the three springs in parallel is, $\delta_e = \delta$.

$$mg = k_1\,\delta + k_2\delta + k_3\delta$$
$$= (k_1 + k_2 + k_3)\,\delta$$
$$mg = k_e\,\delta_e$$
$$(k_1 + k_2 + k_3)\,\delta = k_e\,\delta_e\text{, since }\delta_e = \delta$$
$$k_e = k_1 + k_2 + k_3$$

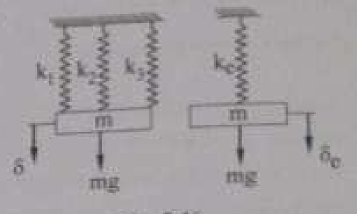

Fig. 5.46

Example 5.42

A 80 N weight is hung on the end of a helical spring and is set vibrating vertically. The weight makes 4 oscillations per second. Determine the stiffness of the spring

Solution.

$$m = \frac{W}{g} = \frac{80}{9.81}$$
$$f = 4 \text{ cps}$$
$$f_n = \frac{1}{2\pi} \times \sqrt{\frac{k}{m}}$$
$$f_n^2 = \frac{1}{4\pi^2} \times \frac{k}{m}$$
$$k = 4\pi^2 f_n^2\, m = 4 \times \pi^2 \times 4^2 \times \frac{80}{9.81}$$
$$= 5151 \text{ N/m}$$

Example 5.43

If a helical spring having a stiffness of 90 N/cm is available, what weight should be hung on it so that it will oscillate with a periodic time of 1 sec.

Solution.

$$k = 90 \text{ N/cm} = 90 \times 100 \text{ N/m.}$$
$$t_p = 1\text{ s}$$
$$f_n = \frac{1}{t_p} = 1 \text{ cps}$$

$$= \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$f_n^2 = \frac{1}{4\pi^2} \times \frac{k}{m}$$

$$m = \frac{k}{4\pi^2 f_n^2} = \frac{90 \times 100}{4 \times \pi^2 \times 1^2} = 227.97 \text{ kg}$$

Example 5.44 [KTU July 2016]

A weight of 50 N suspended from a spring vibrates vertically with an amplitude of 8 cm and a frequency of 1 oscillation per second. Find (a) the stiffness of the spring. (b) the maximum tension induced in the spring and (c) the maximum velocity of the weight.

Solution.

$$m = \frac{W}{g} = \frac{50}{9.81}$$

$$X = 8 \text{ cm} = 0.08 \text{ m}$$

$$f = 1 \text{ cps}$$

(a) $$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$k = f^2 \times 4 \times \pi^2 \times m = 1 \times 4 \times \pi^2 \times \frac{50}{9.81}$$

$$= 201.22 \text{ N/m.}$$

(b) Maximum tension in the spring $= kX$

$$= 201.22 \times 0.08$$

$$= 16.1 \text{ N}$$

(c) Since the body vibrates with S. H. M,

The maximum velocity $= \omega X$

$$= (2\pi f) \times X$$

$$= 2\pi \times 1 \times 0.08$$

$$= 0.5 \text{ m/s}$$

Example 5.45 [KTU Jan 2016]

A body of mass 50 kg is suspended by two springs of stiffness 4 kN/m and 6 kN/m as shown in Fig. 5.47 (a), (b) and (c). The body is pulled 50 mm down from its equilibrium position and then released. Calculate

(i) the frequency of oscillation

(ii) maximum velocity and

(iii) maximum acceleration

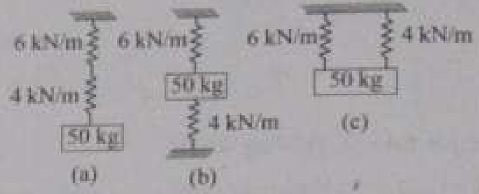

Fig. 5.47

Solution

When the springs are arranged as shown in Fig 5.47 (a)

The equivalent stiffness is given by

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{6} + \frac{1}{4} = \frac{10}{24}$$

$$k_e = \frac{24}{10} = 2.4 \text{ kN/m}$$

$$= 2.4 \times 1000 \text{ N/m}$$

(i) $$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{2.4 \times 1000}{50}}$$

$$= 1.10 \text{ c p s}$$

$$\omega = 2\pi f = 6.93 \text{ rod/s.}$$

(ii) maximum velocity $= \omega X$

$$= 6.93 \times 0.05$$

$$= 0.35 \text{ m/s}$$

(iii) acceleration $= \omega^2 X$

$$= 6.93^2 \times .05$$

$$= 2.4 \text{ m/s}^2$$

(b) When the springs are as shown in Fig 5.47 (b)

The springs are in parallel. Hence the equivalent stiffness;

$$k_e = k_1 + k_2$$

$$= 4 + 6 = 10 \text{ kN/m}$$

(i) Frequency $f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{10 \times 1000}{50}}$

$= 2.25$ c.p.s.

(ii) Maximum velocity $= \omega X$

$= (2\pi f) X$

$= 2\pi \times 2.25 \times 0.05 = 0.71$ m/s

acceleration $= \omega^2 X = (2\pi f)^2 \times 0.05$

$= (2\pi \times 2.25)^2 \times 0.05 = 10 \text{m/s}^2$

When the springs are arranged as shown in Fig. 5.47(c) the springs are in parallel.

Hence the frequency = 2.25 cps.

Maximum velocity = 0.71 m/s and

Acceleration = 10 m/s²

Example 5.46

A spring of stiffness 6kN/m is cut into two halves and fixed to a mass m as shown in Fig. 5.48. If the system vibrates with frequency 3 Hz, determine the mass m.

Solution

Given: Frequency = 3Hz = 3cps

Stiffness of original spring = 6kN/m

Stiffness of spring is inversely proportional to the number of coils. Therefore when this spring is cut into two halves, the stiffness of each half is doubled. $k_1 = k_2 = 2 \times 6 = 12$kN/m. Since the springs are in parallel, the equivalent stiffness, $k_e = k_1 + k_2 = 24$kN/m $= 24 \times 10^3$ N/m.

Frequency, $f = \frac{1}{2\pi}\sqrt{\frac{k_e}{m}}$

$3 = \frac{1}{2\pi}\sqrt{\frac{24 \times 10^3}{m}}$

$9 = \frac{1}{4\pi^2}\frac{24 \times 10^3}{m}$

$m = 67.62$ kg

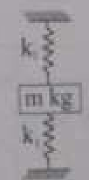

Fig. 5.48

Example 5.47 (KTU May 2019)

A helical spring under a weight of 20N extends by 0.3mm. A weight of 700N is supported on the same spring. Determine the period and frequency of vibration of the spring when they are displaced vertically by a distance of 0.9cm and released. Find the velocity of the weight when the weight is 4mm below its equilibrium position. Take the weight of spring as negligible.

Solution

Given, When $W = 20N$, $\delta = 0.3mm = 0.3\times10^{-3}$ m.

$W = 700N$, $X = 0.9cm$, $x = 4mm = 4\times10^{-3}$ m

To calculate, time period, frequency, velocity when x = 4mm.

We have, $mg = k\delta$

$$\text{Stiffness } k = \frac{mg}{\delta} = \frac{20}{0.3\times10^{-3}} = 66.67\times10^{3} \text{ N/m}$$

$$\text{Time period, } t_p = 2\pi\sqrt{\frac{m}{k}} = 2\pi\times\sqrt{\frac{700}{9.81\times66.67\times10^{3}}} = 0.206 \text{ s}$$

$$\text{Frequency, } f = \frac{1}{t_p} = \frac{1}{0.206} = 4.85 \text{ cps}$$

$$\text{Velocity, } v = \omega\sqrt{X^2 - x^2}$$

$$\omega = 2\pi f = 2\times\pi\times4.85 = 30.47 \text{ rad/s}$$

$$v = 30.47\sqrt{0.9\times10^{-2} - 4\times10^{-3}} = 2.15 \text{ m/s}$$

Example 5.48 (KTU May 2017)

A tray of mass m kg is mounted on springs as shown in Fig. 5.49. The period of vibration of empty tray is 0.5sec. After placing a mass of 1.5kg on the tray, the period was observed to be 0.6sec. Find the mass of the tray and stiffness of each spring.

Solution

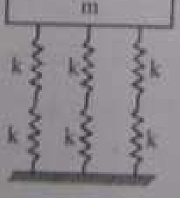

Fig. 5.49

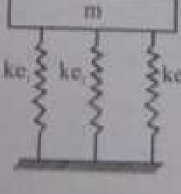

Fig. 5.50

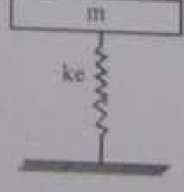

Fig. 5.51

Equivalent stiffness of springs in series as shown in Fig. 5.49 is given by

$$\frac{1}{ke_1} = \frac{1}{k} + \frac{1}{k}$$

$$ke_1 = \frac{k}{2}$$

Equivalent stiffness of the three springs in parallel as shown in Fig. 5.50,

$$ke = ke_1 + ke_1 + ke_1$$

$$= 3ke_1 = \frac{3k}{2}$$

$$\text{Time period, } t_p = 2\pi\sqrt{\frac{m}{ke}} = 2\pi \times \sqrt{\frac{2m}{3k}}$$

$$t_p^2 = 4\pi^2 \times \frac{2m}{3k} = 26.32\,\frac{m}{k}$$

$$\text{In the first case, } 0.5^2 = 26.32\frac{m}{k}$$

$$\text{In the second case, } 0.6^2 = 26.32\frac{m+1.5}{k}$$

$$\text{Dividing the above expressions, } \frac{0.36}{0.25} = \frac{m+1.5}{m}$$

$$1.4\,m = m + 1.5$$

$$m = 3.41 \text{ kg}$$

$$\text{From equation, } 0.5^2 = 26.32 \times \frac{m}{k}$$

$$0.5^2 = 26.32 \times \frac{3.41}{k}$$

$$k = 359 \text{N/m}$$

5.11. Effect of damping

Any resistance to vibratory motion is called damping. In free vibration, damping in its various forms such as air damping (resistance due to air), viscous damping (resistance due to fluid friction), coulomb damping (resistance due to dry friction), magnitic damping (resistance due to magnetic force) etc will always slow down the motion and finally the vibrating body will be brought to rest in its equilibrium position. In viscous damping, the damping force is proportional to the velocity and this constant of proportionality is called coefficient of damping and is denoted by c. Thus damping force, $F_d = c \times$ velocity. It is a resisting force and hence it's direction is opposite to the direction of velocity. Fig. 5.52 shows a spring mass system provided with a viscous damper. The forces acting on the mass are :

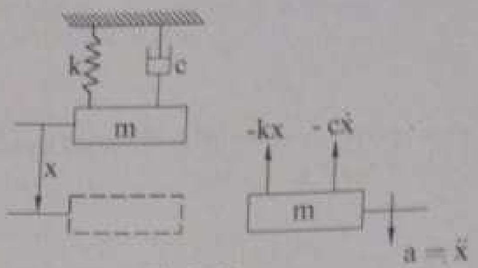

Fig. 5.52

1. Spring force, kx, opposite to the displacement

2. Damping force, $F_d = c\dot{x}$

3. The accelerating force, $F_a = m\ddot{x}$

Net force = mass × acceleration

$$(-kx) + (-c\dot{x}) = m\ddot{x}$$

$m\ddot{x} + c\dot{x} + kx = 0$. This is the equation of motion of damped, free vibration of spring mass system.